Sheet 10

Unless stated otherwise we work over an algebraically closed field k.

1. (Hartshorne Exercise I.6.1) Recall that a curve is *rational* if it is birationally equivalent to $\mathbb{P}^1$. Let Y be a nonsingular rational curve which is not isomorphic to $\mathbb{P}^1$. Show

- **a)** Y is isomorphic to an open subset of $\mathbb{A}^1$.
- **b)** Y is affine.
- **c)** $A(Y)$ is a unique factorization domain.

2. (Hartshorne Exercise I.6.2, *an elliptic curve*) Let Y be the curve $y^2 = x^3 - x$ in $\mathbb{A}^2$ and assume that $\operatorname{char} k \neq 2$. In this exercise we will show that Y is not a rational curve, and hence $K := k(Y)$ is not a pure transcendental extension of k.

- **a)** Show that Y is nonsingular, and deduce that $A := A(Y) \cong k[x,y]/(y^2 - x^3 + x)$ is an integrally closed domain.
- **b)** Let $k[x]$ be the subring of K generated by the image of $x \in A$. Show that $k[x]$ is a polynomial ring, and that A is the integral closure of $k[x]$ in K.
- **c)** Show that there is an automorphism $\sigma : A \to A$ which sends y to $-y$ and leaves x fixed. For any $a \in A$, define the *norm* of a to be $N(a) = a\sigma(a)$. Show that $N(a) \in k[x]$, $N(1) = 1$ and $N(ab) = N(a)N(b)$ for any $a, b \in A$.
- **d)** Using the norm, show that the units in A are precisely the nonzero elements of k. Show that x and y are irreducible elements of A. Show that A is not a unique factorization domain.
- **e)** Using exercise **1**, prove that Y is not a rational curve.

3. (Hartshorne Exercise I.6.6, *Automorphisms of* $\mathbb{P}^1$) Think of $\mathbb{P}^1$ as $\mathbb{A}^1 \cup \{\infty\}$. Then we define a *fractional linear transformation* of $\mathbb{P}^1$ by sending $x \mapsto \frac{ax+b}{cx+d}$, for $a, b, c, d \in k$, $ad - bc \neq 0$.

a) Show that a fractional linear transformation induces an *automorphism* of $\mathbb{P}^1$ (i.e., an isomorphism of $\mathbb{P}^1$ with itself). We denote the group of all these fractional linear transformations by PGL(2).

b) Let $\operatorname{Aut}\mathbb{P}^1$ denote the group of all automorphisms of $\mathbb{P}^1$. Show that $\operatorname{Aut}\mathbb{P}^1 \cong \operatorname{Aut}k(x)$, the group of k-automorphisms of the field $k(x)$.

c) Now show that every automorphism of $k(x)$ is a fractional linear transformation, and deduce that $\mathrm{PGL}(2) \to \operatorname{Aut}\mathbb{P}^1$ is an isomorphism.

d) Given three distinct points z_1, z_2, z_3 and w_1, w_2, w_3 on $\mathbb{P}^1$, there is a unique $\varphi \in \mathrm{PGL}(2)$ such that $\varphi(z_1) = w_1$, $\varphi(z_2) = w_2$, $\varphi(z_3) = \varphi(w_3)$.

4. (Harshorne Exercise I.6.7) Let $P_1, \ldots, P_r, Q_1, \ldots, Q_s$ be distinct points in $\mathbb{A}^1$. If $\mathbb{A}^1 \setminus \{P_1, \ldots, P_r\}$ is isomorphic to $\mathbb{A}^1 \setminus \{Q_1, \ldots, Q_s\}$, show that $r = s$. Is the converse true?

Hint: Exercise **3**

5. (*Elliptic and hyperelliptic curves*) Let $\operatorname{char} k = 0$. Let $p \in k[x]$ be a nonconstant polynomial of degreee d with simple zeros. Then $C = V(y^2 - p(x)) \subseteq \mathbb{A}^2$ is a smooth curve, cf. sheet 8, exercise **4a**. Let $\overline{C}$ be its projective closure in $\mathbb{P}^2$. Let $\widetilde{C}$ be the nonsingular compactification of C. For $d \in \{3, 4\}$ we call $\widetilde{C}$ an *elliptic curve*, for $d \geq 5$ a *hyperelliptic curve*. The morphism $f : C \to \mathbb{A}^1$, $(x, y) \mapsto x$, extends to $\widetilde{f} : \widetilde{C} \to \mathbb{P}^1$. Show

a) f has ramification index $e_{(x,y)}(f) = 1$ if $p(x) \neq 0$ and $e_{(x,y)}(f) = 2$ if $p(x) = 0$.

b) If $d \leq 2$, then $\overline{C} \cong \widetilde{C} \cong \mathbb{P}^1$.

c) If $d \geq 3$, then $C_\infty = \overline{C} \setminus C$ is a single point. It is a singular point of $\overline{C}$ if and only if $d \geq 4$.

Hint: Jacobian criterion

d) If $d \geq 3$, $\widetilde{C}$ can be described as C glued with

$$U = \left\{ (u, v) \in \mathbb{A}^2 \;\middle|\; v^2 = u^{2\lfloor \frac{d+1}{2} \rfloor} p\left(\frac{1}{u}\right) \right\}$$

by the isomorphism

$$C \cap D(x) \to U \cap D(u) \,,\, (x, y) \mapsto \left(x^{-1}, yx^{-\lfloor \frac{d+1}{2} \rfloor}\right) \,.$$

Conclude that $\widetilde{C} \setminus C = \widetilde{f}^{-1}(\infty)$ consists of one or two points depending on whether d is odd or even.

e) Let $\widetilde{C}$ be an elliptic curve. Then $\widetilde{C}$ is isomorphic to a $\widetilde{C}$ given by $p(x) = x(x-1)(x-\lambda)$ for some $\lambda \in k \setminus \{0, 1\}$.

Hint: Exercise **3d**

6. Let $\varphi : C_1 \to C_2$ be a morphism of nonsingular projective curves that is not constant. Let $\varphi^\sharp : k(C_2) \to k(C_1)$ be the corresponding embedding of function fields. We have a pullback map $\varphi^* : \mathrm{Div}(C_2) \to \mathrm{Div}(C_1)$. Let $\varphi_* : \mathrm{Div}(C_1) \to \mathrm{Div}(C_2)$ be the pushforward map. Show

a) $\varphi_*\varphi^*$ is multiplication by $\deg \varphi$ on $\mathrm{Div}(C_2)$.

b) $\varphi^*\varphi_*$ is not multiplication by $\deg \varphi$ on $\mathrm{Div}(C_1)$ in general, but we have $\deg(\varphi^*\varphi_* D) = \deg \varphi \deg D$ for all $D \in \mathrm{Div}(C_1)$.

c) $\nu_p(\varphi^\sharp f) = e_p(\varphi)\nu_{\varphi(p)}(f)$ for all $p \in C_1$ and $f \in k(C_2)^\times$. Then show $\varphi^*(\mathrm{div}\, f) = \mathrm{div}(\varphi^\sharp\, f)$ for $f \in k(C_2)^\times$.

d) The sequence

$$1 \to k^\times \to k(\mathbb{P}^1)^\times \xrightarrow{\mathrm{div}} \mathrm{Div}(\mathbb{P}^1) \xrightarrow{\deg} \mathbb{Z} \to 0$$

is exact.

Due on Friday, May 22.