Sheet 11

Unless stated otherwise we work over an algebraically closed field k.

1. Let C be a smooth projective curve and D a divisor on C. Show

a) The assignment $f \mapsto \operatorname{div} f + D$ induces a bijection between $(\mathcal{L}(D) \setminus \{0\})/k^\times$ and the set of effective divisors whose image in $\operatorname{Pic}(C)$ coincides with that of D.

b) $\mathcal{L}(D) \neq 0$ implies $\deg D \geq 0$.

c) Assume $\deg D = 0$. Then $\mathcal{L}(D) \neq 0$ holds if and only if $D = 0$ holds in $\operatorname{Pic}(C)$.

2. Let $F \in k[x, y]$ be a cubic polynomial. If $F = y^2 - x^3 - ax^2 - bx - c$ for some $a, b, c \in k$ we say that F has *Weierstrass normal form.* Show that if $\operatorname{char} k \notin \{2, 3\}$, then $x = \frac{12c}{u+v}$ and $y = 36c\frac{u-v}{u+v}$ define a birational equivalence between $V(u^3 + v^3 - c) \subseteq \mathbb{A}^2$ and $V(y^2 - x^3 + 432c^2) \subseteq \mathbb{A}^2$.

Remark: If $\operatorname{char} k \neq 2$, any $V(F) \subseteq \mathbb{A}^2$ can be shown to be birationally equivalent to a $V(\widetilde{F})$, where $\widetilde{F}$ has Weierstrass normal form.

3. Let $\operatorname{char} k \neq 2$. Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be points in $V(F) \subseteq \mathbb{A}^2$, where $F = y^2 - x^3 - ax^2 - bx - c$ is such that $V(F)$ is smooth. Show that $P_1 \oplus P_2 = (x_3, -y_3)$ is given by

a)

$$x_3 = \lambda^2 - a - x_1 - x_2 \quad y_3 = \lambda x_3 + \nu \quad \lambda := \frac{y_2 - y_1}{x_2 - x_1} \quad \nu := y_1 - \lambda x_1 = y_2 - \lambda x_2$$

if P_1 and P_2 do not lie on a vertical line in $\mathbb{A}^2$.

b) by the formula of **a** with $\lambda = \frac{3x_1^2 + 2ax_1 + b}{2y_1}$ if $P_1 = P_2$ does not lie on the x-axis.

c) What happens if P_1 and P_2 are on a vertical line or $P_1 = P_2$ lie on the x-axis?

4. Let $\operatorname{char} k \notin \{2, 3\}$. Let $C = \overline{V(F)} \subseteq \mathbb{P}^2$ be nonsingular, where $F = y^2 - x^3 - ax^2 - bx - c$. Show that C has exactly nine points P of order dividing three, i.e. satisfying $P \oplus P \oplus P = 0$, and that they form a subgroup of C isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. Show that these nine points are the *inflection points of* C, i.e. for each P there is a line $V(\lambda)$ such that $v_P(\lambda) = 3$.

Due on Friday, May 29.