Sheet 1

Unless stated otherwise k denotes an algebraically closed field.

1. Consider the subsets Y of $\mathbb{A}^2$ given below. Determine

- the ideal $I(Y)$ in $\mathrm{k}[x,y]$ of polynomials vanishing on Y.
- the closure of Y in $\mathbb{A}^2$ w.r.t. the Zariski topology.
- whether Y is an affine variety in $\mathbb{A}^2$.

a) $(\{0\} \times \mathbb{A}^1) \cup (\mathbb{A}^1 \times \{0\})$

b) the graph of $t \mapsto e^t$, $\mathrm{k} = \mathbb{C}$

c) $\{1\} \times \mathbb{Z}$

d) $\mathbb{Z} \times \mathbb{Z}$

e) $\{(t^m, t^n) \mid t \in \mathrm{k}\}$ for given $m, n \in \mathbb{Z}_{\geq 1}$

f) $V(\{x^2 + y - 1, y^2\})$

2. * Show

a) $\mathrm{SL}_2(\mathbb{C})$ is an affine variety in $\mathrm{Mat}_2(\mathbb{C}) = \mathbb{A}^4$.

b) $\mathrm{SL}_2(\mathbb{Z})$ is dense in $\mathrm{SL}_2(\mathbb{C})$ w.r.t. the Zariski topology.

Hint: You may use without proof that any element of $\mathrm{SL}_2(\mathbb{C})$ can written as a multiple product of matrices of the form $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ with $b, c \in \mathbb{C}$. Use a generalization of the result of exercise **1d**.

3. (*Conics in* $\mathbb{A}^2$, Hartshorne Exercise I.1.1)

a) Show that for $Y = V(y - x^2)$ in $\mathbb{A}^2$ the ring of regular functions $A(Y)$ is isomorphic to $\mathrm{k}[t]$.

b) Show that for $Y = V(xy - 1)$ in $\mathbb{A}^2$ $A(Y)$ is isomorphic to $\mathrm{k}[t, t^{-1}]$.

c) Let $f \in \mathrm{k}[x,y]$ be an irreducible quadratic polynomial. Show that for $Y = V(f)$ $A(Y)$ is isomorphic to $A(Y)$ from **a** or **b** and give the corresponding explicit condition on f.

d) What are the additional possibilities in **c** if we drop the condition that f be irreducible?

4. Let $f_1, f_2 \in \mathrm{k}[x_1, \ldots, x_n]$ be such that each irreducible factor of f_1 and f_2 occurs only once. Show that $V(f_1) = V(f_2)$ implies $f_1 = cf_2$ for some $c \in \mathrm{k}^\times = \mathrm{k} \setminus \{0\}$.

5. * (Gathmann lecture notes Exercise 1.9) Prove that every affine variety in $\mathbb{A}^n$ consisting of only finitely many points can be written as the zero locus of n polynomials.

6. (Hartshorne Exercise I.3.2a) Show that $\mathbb{A}^1 \to \mathbb{A}^2$, $t \mapsto (t^2, t^3)$, is a homeomorphism onto $Y = V(y^2 - x^3)$, but $k[t] = A(\mathbb{A}^1)$ is not isomorphic to $A(Y)$.

Due on Friday, February 27.