Sheet 2

Unless stated otherwise k denotes an algebraically closed field.

1. Exercise 3 in the lecture notes.

2. Let $Y \subseteq \mathbb{A}^n$ be an affine variety. Show

a) If $Y = Y_1 \sqcup Y_2$ (disjoint union), where Y_1 and Y_2 are nonempty affine varieties, then $A(Y) \cong A(Y_1) \times A(Y_2)$ as k-algebras, where $\times$ is the product of k-algebras.

b) Y is connected if and only if $f^2 = f$ in $A(Y)$ implies $f \in \{0, 1\}$.

Definition: A topological space X is *connected* if $X = X_1 \sqcup X_2$, where X_1 and X_2 are closed (equivalently open) subsets, implies X_1 or X_2 is X.

3. Consider the subspaces Y of $\mathbb{A}^2$ given in sheet 1, exercise **1**. Determine

a) whether $\overline{Y}$ is connected.

b) the irreducible components of $\overline{Y}$.

Remark: A topological space X is *irreducible* if $X = X_1 \cup X_2$, where X_1 and X_2 are distinct closed subsets of X, implies X_1 or X_2 is X. According to Hartshorne, Proposition I.1.5, there is a unique decomposition $X = \overline{Y} = \bigcup_{j=1}^{r} X_j$ of X into finitely many irreducible subspaces X_j such that $X_i \not\subseteq X_j$ for all $i \neq j$. The X_j are the *irreducible components* of X.

4. Exercise 7 in the lecture notes.

5. (*Products of Affine Varieties*, Hartshorne Exercise I.3.15) Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine varieties. Show

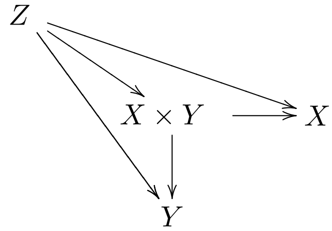
a) that if X and Y are irreducible, then $X \times Y \subseteq \mathbb{A}^{n+m}$ with its induced topology is irreducible. The affine variety $X \times Y$ is called the *product* of X and Y. Note that its topology is in general not equal to the product topology.

Hint: Suppose that $X \times Y$ is a union of two closed subsets $Z_1 \cup Z_2$. Let $X_i = \{x \in X \mid x \times Y \subseteq Z_i\}$, $i = 1, 2$. Show that $X = X_1 \cup X_2$ and X_1, X_2 are closed. Then $X = X_1$ or X_2 so $X \times Y = Z_1$ or Z_2.

b) $A(X \times Y) \cong A(X) \otimes_{\mathrm{k}} A(Y)$ as k-algebras

c) $X \times Y$ is a product in the category of varieties, i.e.

1. the projections $X \times Y \to X$ and $X \times Y \to Y$ are morphisms.
2. given a variety Z and the morphisms $Z \to X$, $Z \to Y$, there is a unique morphism $Z \to X \times Y$ such that the diagram



commutes.

6. Exercise 12 in the lecture notes.

7. Exercise 13 in the lecture notes.

Due on Friday, March 6.