Sheet 3

Unless stated otherwise k denotes an algebraically closed field.

1. Consider $Y = V(\{x^2 - y^3\}) \subseteq \mathbb{A}^2$ and the function $f : Y \to \mathbb{A}^1 = k$ given by

$$f(x,y) = \begin{cases} \frac{x}{y} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}.$$

Show that

a) f is continuous.

b) f is not a regular function.

c) f restricts to a regular function on $Y_y = Y_x$.

2.

1. (lecture notes) Let $X := \mathbb{A}^2 := \operatorname{Specm} k[x_1, x_2]$ and $U := \mathbb{A}^2 - \{(0,0)\}$. Show that the restriction map $A(X) = \mathcal{O}(X) \to \mathcal{O}(U)$ is an isomorphism.
2. (lecture notes) Show that the quasi-affine variety $U := \mathbb{A}^2 - \{(0,0)\}$ is not isomorphic to any affine variety.

3. (lecture notes) Let X be an affine variety with affine coordinate ring $A := A(X)$, let $a \in A$, let U be an open subset of $D_X(a)$. Recall that $\iota_{X,a} : X_a \to X$ is the morphism of affine varieties with image $D_X(a)$ corresponding to the localization-at-a map $\iota^\sharp_{X,a} : A \to A(X_a) = A_a$ of affine coordinate rings. Set $V := \iota^{-1}_{X,a}(U)$. Show that a function $f : U \to k$ belongs to $\mathcal{O}(U)$ if and only if $f|_{X,a} := f \circ \iota_{X,a}$ belongs to $\mathcal{O}(V)$.

4. (lecture notes)

a) Verify for a quasi-affine variety X that morphisms $X \to \mathbb{A}^1$ are the same as regular functions $X \to k$.

b) * Verify for an *affine* variety X and a quasi-affine variety Y that morphisms of quasi-affine varieties

$$f : Y \to X$$

are in bijection with k-algebra morphisms

$$\varphi : A(X) \to \mathcal{O}(Y)$$

under the mutually inverse maps $f \mapsto f^\sharp$ and $\varphi \mapsto \varphi^\flat$, where $f^\sharp(a) := a \circ f$ and $\varphi^\flat(\beta) := \mathrm{pt}(\mathrm{eval}_\beta \circ f)$ for $a \in A(X), \beta \in Y$.

c) Explain how the second part of this exercise specializes to the first part.

5. (lecture notes) Let $(X, \mathcal{O}_X)$ be a prevariety and $\alpha \in X$. Verify the following assertions:

a) The neighborhoods U of α form a directed set with respect to containment. The *stalk of* $\mathcal{O}_X$ *at* α is defined to be

$$\mathcal{O}_{X,\alpha} := \varinjlim_{U \ni \alpha} \mathcal{O}_X(U).$$

It is naturally a k-algebra.

b) For any neighborhood U of α, the evaluation-at-α morphism $\mathrm{eval}_\alpha : \mathcal{O}_X(U) \to k$ induces a morphism $\mathcal{O}_{X,\alpha} \to k$. We denote the latter also by eval_α.

c) The k-algebra $\mathcal{O}_{X,\alpha}$ is a local ring. Its unique maximal ideal is the kernel $\mathfrak{m}_\alpha$ of $\mathrm{eval}_\alpha : \mathcal{O}_{X,\alpha} \to k$, and we have $\mathcal{O}_{X,\alpha}/\mathfrak{m}_\alpha = k$ as k-algebras.

d) For any affine neighborhood Y of α, the open neighborhoods $D_Y(a)$ of α (taken over $a \in A(Y)$ with $a(\alpha) \neq 0$) form a directed set that is cofinal in the the directed set of all neighborhoods of α. The restriction maps $\mathcal{O}_X(U) \to \mathcal{O}_X(D_Y(a))$ for $\alpha \in D_Y(a) \subset U$ thereby induce an identification

$$\mathcal{O}_{X,\alpha} = \varinjlim_{D_Y(a) \ni \alpha} \mathcal{O}_X(D_Y(a)) = \varinjlim_{D_Y(a) \ni \alpha} A(Y)_a \cong A(Y)_\mathfrak{m}$$

where $\mathfrak{m}$ is the maximal ideal $\mathfrak{m} := \{a \in A(Y) : a(\alpha) = 0\}$ of $A(Y)$.

6. (cf. Hartshorne Exercise I.3.3) Let Y and X be quasi-affine varieties and $f : Y \to X$ be a morphism. Let $\beta \in Y$. Show

a) For $U \subseteq X$ open the maps $f_U^\sharp : \mathcal{O}_X(U) \to \mathcal{O}_Y(f^{-1}(U))$ induce a map $f_\beta^\sharp : \mathcal{O}_{X,f(\beta)} \to \mathcal{O}_{Y,\beta}$ of k-algebras such that $(f_\beta^\sharp)^{-1}(\mathfrak{m}_\beta) = \mathfrak{m}_{f(\beta)}$.

b) f is an isomorphism if and only if f is a homeomorphism and $f_\beta^\sharp$ is an isomorphism for all β.

c) If Y is irreducible and $f(Y)$ is dense in X, then $f_\beta^\sharp$ is injective for all β.

Due on Friday, March 13.