Sheet 4

In the following exercises we always work over an algebraically closed field.

1. Let X be the projective variety $V(x_0x_2^2 - x_1^3 + x_0^2x_1) \subset \mathbb{P}^2$. For each $i \in \{0, 1, 2\}$, compute the affine variety $X_i \subset \mathbb{P}^2_i \cong \mathbb{A}^2$ (notation as in lecture) obtained by dehomogenizing.

2. *(d-uple or Veronese embedding, cf. Hartshorne Exercise I.2.12)* Let $n, d \in \mathbb{Z}_{>0}$. Consider the monomials of degree d in the $n+1$ variables $x_0, \ldots, x_n$, i.e. elements of the form $x^i := x_0^{i_0} \ldots x_n^{i_n}$ where $i \in \mathbb{Z}_{\geq 0}^{n+1}$ is a multi-index such that $i_0 + \cdots + i_n = d$. We define $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$ by $[x_0, \ldots, x_n] \mapsto [(x^i)_i]$ called *d-uple or Veronese embedding* of $\mathbb{P}^n$ in $\mathbb{P}^N$. For example, if $n = 1$, $d = 2$, then $N = 2$, and the image of the 2-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$ is a conic. Show

a) $N = \binom{n+d}{n} - 1$

b) Let $\theta : k[(y_i)_i] \to k[x_0, \ldots, x_n]$ be the homomorphism defined by sending $y_i \mapsto x^i$ and let $\mathfrak{a}$ be the kernel of θ. Then $\mathfrak{a}$ is a homogeneous prime ideal and so $V(\mathfrak{a})$ is an irreducible projective variety in $\mathbb{P}^N$.

c) The image of ν_d is $V(\mathfrak{a})$.

d) ν_d is a homeomorphism of $\mathbb{P}^n$ onto $V(\mathfrak{a})$.

e) * The ideal $\mathfrak{a}$ is generated by

$$y_i y_j - y_k y_l \quad i + j = k + l\ .$$

3. *(Harris Exercise 1.3)* Let Γ be a finite subset of $\mathbb{P}^n$ of cardinality $|\Gamma| = d$. Show that if Γ is not contained in a *line in* $\mathbb{P}^n$, i.e. a set $V(f_1, \ldots, f_{n-1})$, where $f_j \in k[x_0, \ldots, x_n]_1$ are linearly independent, then Γ may be described as the zero locus of polynomials of degree $d - 1$ or less.

Hint: Induction on d. The induction start is $d = 3$.

4. * *(Harris Example 1.2)* Let Γ be a finite subset of $\mathbb{P}^n$ of cardinality $|\Gamma| = d$. We say that Γ is *in general position* if any subset of Γ whose lift to $\mathbb{A}^{n+1}$ is linearly dependent has cardinality $> n+1$. (If $d \geq n+1$ this is the same as saying that the lift of any $n+1$ points of Γ do not lie in a hyperplane in $\mathbb{A}^{n+1}$.) Show that if $d \leq 2n$ and Γ is in general position, then Γ may be described as the zero set of quadratic polynomials.

Hint: Read the proof for $d = 2n$ in Harris.

5. *(Rational normal curve)* The image of $\nu_d : \mathbb{P}^1 \to \mathbb{P}^d$ composed with an element of the projective linear group $\mathrm{PGL}_{d+1}(k)$ is called a *rational normal curve* C. In the special case $d = 3$ it is called *twisted cubic curve*. Show

a) Any $d+1$ distinct points on C are in general position.

Hint: Vandermonde determinant

b) *(Harris Example 1.17)* Let $[\mu_i, \nu_i] \in \mathbb{P}^1$, $1 \leq i \leq d+1$, be $d+1$ distinct points. Set $H_i(x_0, x_1) = \prod_{j \neq i}(\mu_j x_0 - \nu_j x_1)$. Then $[x_0, x_1] \mapsto [(H_i(x_0, x_1))_{1 \leq i \leq d+1}]$ is a parametrization of a rational normal curve sending $[\nu_i, \mu_i]$ to the image of the ith standard basis vector in $\mathbb{P}^d$.

c) *(Harris Theorem 1.18)* If Γ is a subset in $\mathbb{P}^d$ of cardinality $d+3$ and in general position, there is a unique rational normal normal curve passing through Γ.

Hint: For the existence determine the image of $[0,1]$ and $[1,0]$ in the parametrization in **b**.

6. *(Projection from a point, Harris Example 3.4)* Let $\mathbb{P}^{n-1} \subseteq \mathbb{P}^n$ be a hyperplane and $p \in \mathbb{P}^n - \mathbb{P}^{n-1}$. Define

$$\pi_p : \mathbb{P}^n - \{p\} \to \mathbb{P}^{n-1}, \ q \mapsto \mathbb{P}^{n-1} \cap \text{line through } p \text{ and } q$$

and call it *projection from the point p to the hyperplane* $\mathbb{P}^{n-1}$. We can choose homogeneous coordinates on $\mathbb{P}^n$ such that π_p is given by $[x_0, \ldots, x_n] \mapsto [x_0, \ldots, x_{n-1}]$.

a) Verify that π_p is a morphism.

b) Set $n = 3$. Find the equations of $\pi_p(C)$ for C the twisted cubic and $p = [1,0,0,1]$ and $p = [0,1,0,0]$.

c) Show that if C is a rational normal curve in $\mathbb{P}^n$ and $p \in C$, then $\overline{\pi_p(C - p)}$ is a rational normal curve in $\mathbb{P}^{n-1}$.

Due on Friday, March 20.