Sheet 5

Unless stated otherwise we work over an algebraically closed field k.

1. The *affine Veronese surface* $S \subseteq \mathbb{A}^5$ is the image of $\varphi : \mathbb{A}^2 \to \mathbb{A}^5$ given by $\varphi(x_1, x_2) = (x_1, x_2, x_1^2, x_1x_2, x_2^2)$. The projective closure $\overline{S} \subseteq \mathbb{P}^5$ is known as the *projective Veronese surface.*

a) Find a set of homogeneous equations for $\overline{S}$.

Hint: E.g. you can determine a Gröbner basis for $I(S)$ using the Buchberger algorithm.

b) Show that the parametrization of the affine Veronese surface above can be extended to a morphism $\mathbb{P}^2 \to \mathbb{P}^5$ whose image coincides with $\overline{S}$.

2. Let $X \subseteq \mathbb{P}^n$ be a projective variety that is not a finite collection of points. Let $g \in k[x_0, \dots, x_n]_d$ for some $d > 0$. Prove that $V(g) \cap X \neq \varnothing$.

Hint: Either assume $V(g) \cap X = \varnothing$ and construct a nonconstant regular function on some connected component of X or use the Veronese embedding ν_d of sheet 4, exercise **2**.

3. Prove that every rational map $\mathbb{P}^1 \dashrightarrow \mathbb{P}^n$ is regular.

4. *(Harris Exercise 7.13)* Give an explicit birational equivalence of $\mathbb{P}^m \times \mathbb{P}^n$ with $\mathbb{P}^{m+n}$.

5. *(Harris Exercise 7.14)* Let $Q \subseteq \mathbb{P}^n$ be a *quadric*, i.e. the zero locus of a homogeneous polynomial of degree two. Let $p \in Q$ be any point not lying on the vertex of Q. Show that the projection π_p from the point p defined in sheet 4, exercise **6**, defines a birational equivalence $Q \dashrightarrow \mathbb{P}^{n-1}$.

6. *(Hartshorne Exercise I.4.4)* A variety Y is *rational* if it is birationally equivalent to $\mathbb{P}^n$ for some n. Show

a) Any conic in $\mathbb{P}^2$ is a rational curve.

b) The cuspidal cubic $y^2 = x^3$ is a rational curve.

c) Let Y be the nodal cubic curve $x_1^2 x_2 = x_0^2(x_0 + x_2)$ in $\mathbb{P}^2$. The projection π_p from the point $p = [0, 0, 1]$ to the line $x_2 = 0$ induces a birational map from Y to $\mathbb{P}^1$. Thus Y is a rational curve.

Due on Thursday, April 2.