Sheet 7

1. *(Convex polyhedral cones and their duals)* Let V be a finite-dimensional $\mathbb{R}$-vector space and L be a lattice in V. We call a subset $\sigma \subseteq V$ a *(convex polyhedral) cone in* (V, L) if there are $v_j \in L$ such that $\sigma = \sigma(v_1, \dots, v_N) = \sum_{j=1}^N \mathbb{R}_{\geq 0} v_j$. The *dimension* $\dim \sigma$ of σ is defined as $\dim \operatorname{span}_{1 \leq j \leq N} v_j$. Consider a cone σ in $(\mathbb{R}^d, \mathbb{Z}^d)$.

a) Show that $\sigma^* := \{u \in \mathbb{R}^{d*} \mid u(v) \geq 0 \ \forall v \in \sigma\}$ is a cone in $(\mathbb{R}^{d*}, \mathbb{Z}^{d*})$ called the *dual of* σ.

b) σ is called *strongly convex* if it does not contain a line through the origin. Show that this is equivalent to each of the following

1. $\sigma \cap (-\sigma) = \{0\}$
2. $\dim \sigma^* = d$
3. $\{0\}$ is a face of σ.

c) Show that the set $S_\sigma := \sigma^* \cap \mathbb{Z}^{d*}$ is a *submonoid* of $(\mathbb{Z}^{d*}, +)$, i.e. is closed under $+$, and contains 0.

d) Show that the monoid S_σ is finitely generated.

e) A *face of* σ is a set of the form $\tau = \{v \in \sigma \mid u(v) = 0\} = \sigma \cap u^\perp$ for some $u \in S_\sigma$. Then τ is again a cone in $(\mathbb{R}^d, \mathbb{Z}^d)$. Show $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-u)$.

f) * Let σ' be another cone in $(\mathbb{R}^d, \mathbb{Z}^d)$ such that $\tau := \sigma \cap \sigma'$ is a face of σ and σ'. Show that there is a $u \in \sigma^* \cap (-\sigma')^* \cap \mathbb{Z}^{d*}$ with $\tau = \sigma \cap u^\perp = \sigma' \cap u^\perp$. Conclude $S_\tau = S_\sigma + S_{\sigma'}$.

For the remaining exercises σ will denote a cone in $(\mathbb{R}^d, \mathbb{Z}^d)$ that is assumed to be strongly convex.

2. *(Affine toric variety from a cone)* Show

a) Set $\mathbb{C}[x, x^{-1}] := \mathbb{C}[x_1, \dots, x_d, x_1^{-1}, \dots, x_d^{-1}]$ and

$$A_\sigma := \mathbb{C}[S_\sigma] := \left\{ \sum_{\text{finite}} c_u x^u \in \mathbb{C}[x, x^{-1}] \;\middle|\; c_u \in \mathbb{C}, u \in S_\sigma \right\},$$

where $x^u := x_1^{u(e_1)} \dots x_d^{u(e_d)}$. Here $e_1, \dots, e_d$ denote the standard basis of the lattice $\mathbb{Z}^d$ in $\mathbb{R}^d$. Then A_σ is an integral domain and a finitely generated $\mathbb{C}$-algebra generated by monomials. We call $X_\sigma := \operatorname{Specm} A_\sigma$ the *(affine) toric variety associated to* σ.

b) $\mathrm{Hom}_{\mathrm{monoid}}(S_\sigma, \mathbb{C})$, where $\mathbb{C}$ is considered as a monoid under multiplication, is naturally in bijection with X_σ.

c) Let $\tau = \sigma \cap u^\perp$ be a face of σ. Then A_τ identifies naturally with the localization $(A_\sigma)_{x^u}$. Consequently we obtain an open embedding $\iota_{\tau,\sigma} : X_\tau \hookrightarrow X_\sigma$.

d) * Let $u_1, \ldots, u_k$ be generators of S_σ. To the relations $\sum_{j=1}^k \mu_j u_j = \sum_{j=1}^k \nu_j u_j$ in S_σ, where $\mu_j, \nu_j \geq 0$, we associate the ideal I_σ generated by $\xi^\mu - \xi^\nu$ in $\mathbb{C}[\xi] = \mathbb{C}[\xi_1, \ldots, \xi_k]$. Then the assignment $\xi_j \mapsto x^{u_j}$ induces an isomorphism $X_\sigma \xrightarrow{\cong} V(I_\sigma)$.

3. Determine σ^*, generators of S_σ and I_σ and $V(I_\sigma)$ in each of the following cases.

a) $\sigma = \{0\}$ in $(\mathbb{R}^d, \mathbb{Z}^d)$

b) $\sigma = \sigma(v_1, v_2)$ in $(\mathbb{R}^2, \mathbb{Z}^2)$, where (v_1, v_2) is any basis of the lattice $\mathbb{Z}^2$.

c) $\sigma = \sigma(e_1)$ in $(\mathbb{R}^2, \mathbb{Z}^2)$

d) $\sigma = \sigma(2e_1 - e_2, e_2)$ in $(\mathbb{R}^2, \mathbb{Z}^2)$.

4. *(Torus action on an affine toric variety)* Show

a) The *d-dimensional torus*

$$\mathbb{T}^d := X_{\{0\}} = \mathrm{Specm}\, \mathbb{C}[\mathbb{Z}^{d*}] = \mathrm{Specm}\, \mathbb{C}[x, x^{-1}] = (\mathbb{C}^\times)^d$$

is an *affine group variety*, i.e. $\mathbb{T}^d$ is a group and multiplication $m : \mathbb{T}^d \times \mathbb{T}^d \to \mathbb{T}^d$ and inverse map $I : \mathbb{T}^d \to \mathbb{T}^d$ are morphisms. In fact $m^\sharp : \mathbb{C}[x, x^{-1}] \to \mathbb{C}[x, x^{-1}] \otimes_\mathbb{C} \mathbb{C}[x, x^{-1}]$ is given by $f \mapsto f \otimes f$.

b) Under the identification $X_\sigma \cong \mathrm{Hom}_{\mathrm{monoid}}(S_\sigma, \mathbb{C})$

$$\mathrm{ac} : \mathbb{T}^d \times X_\sigma \to X_\sigma,\ (s, t) \mapsto (u \mapsto s(u)t(u))\ ,$$

defines an *action* of $\mathbb{T}^d$ on X_σ, i.e. ac satisfies

$$\mathrm{ac}(1, \alpha) = \alpha\ ,\ \mathrm{ac}(st, \alpha) = \mathrm{ac}(s, \mathrm{ac}(t, \alpha))$$

and ac is a morphism. In fact $\mathrm{ac}^\sharp : A_\sigma \to \mathbb{C}[x, x^{-1}] \otimes_\mathbb{C} A_\sigma$ is given by $f \mapsto f \otimes f$. Thus $\iota_{\{0\},\sigma} : \mathbb{T}^d \hookrightarrow X_\sigma$ respects the $\mathbb{T}^d$-action if we let $\mathbb{T}^d$ act on itself by multiplication. Also note $\dim X_\sigma = d$.

c) * Formulate the action of $\mathbb{T}^d$ on $V(I_\sigma)$.

d) * Check in the examples of **3** that the orbits of the $\mathbb{T}^d$-action on X_σ are naturally in bijection with the faces of σ. Which faces correspond to $\mathbb{T}^d$-fixed points?

5. *(Smoothness of X_σ in terms of σ)* Show

a) If $v_1, \ldots, v_N \in \mathbb{Z}^d$ are part of a $\mathbb{Z}$-basis of $\mathbb{Z}^d$, then the toric variety of the cone $\sigma = \sigma(v_1, \ldots, v_N)$ is $X_\sigma \cong \mathbb{A}^N \times \mathbb{T}^{d-N}$. In particular X_σ is smooth.

b) * If X_σ is smooth, then $\sigma = \sigma(v_1, \ldots, v_N)$ for some $v_1, \ldots, v_N$ that are part of a $\mathbb{Z}$-basis of $\mathbb{Z}^d$.

Hint: Consider the cotangent space $\mathfrak{m}_{x_\sigma}/\mathfrak{m}_{x_\sigma}^2$ at the point $x_\sigma \in X_\sigma$ defined by the monoid homomorphism $S_\sigma \to \mathbb{C}$ given by

$$u \mapsto \begin{cases} 1 & u \in \sigma^\perp \\ 0 & \text{else} \end{cases} .$$

6. * *(Toric variety from a fan)* Let Δ be a *fan*, i.e. a finite collection of cones such that

1. Each face of a cone in Δ is again a cone in Δ.
2. The intersection of two cones in Δ is a face of each.

Show

a) Let X_Δ be the prevariety glued from the X_σ via the open embeddings $\iota_{\sigma\cap\sigma',\sigma} : X_{\sigma\cap\sigma'} \hookrightarrow X_\sigma$ and $\iota_{\sigma\cap\sigma',\sigma'} : X_{\sigma\cap\sigma'} \hookrightarrow X_{\sigma'}$ for $\sigma, \sigma' \in \Delta$. The diagonal map $X_{\sigma\cap\sigma'} \to X_\sigma \times X_{\sigma'}$ is a closed embedding and consequently X_Δ is separated. We call X_Δ the *toric variety associated to* Δ.

Remark: We may take Δ to consist of all the faces of a single cone σ, in which case $X_\Delta = X_\sigma$ holds. Using exercise **4** one can show that there is a $\mathbb{T}^d$-action on X_Δ and an open dense embedding $\mathbb{T}^d \hookrightarrow X_\Delta$ respecting the $\mathbb{T}^d$-action.

b) Find a fan Δ such that $X_\Delta \cong \mathbb{A}^1, \mathbb{P}^1, \mathbb{A}^1 \times \mathbb{P}^1, \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ respectively. Describe the cones $\sigma \in \Delta$ and the corresponding A_σ.

Due on Friday, April 24.