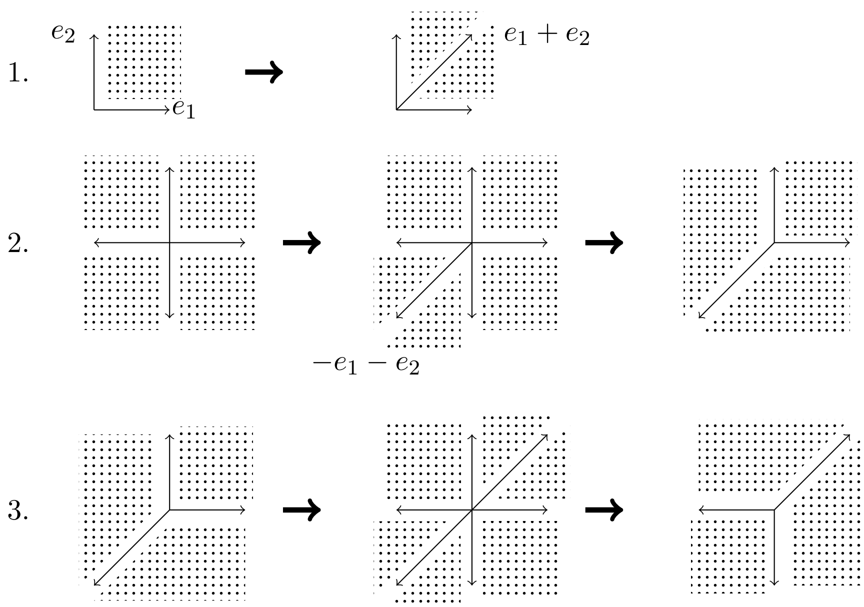
Sheet 8

Unless stated otherwise we work over an algebraically closed field k.

1. Let us explain how the pictures of fans in $(\mathbb{R}^2, \mathbb{Z}^2)$



describe

1. the blow-up of $\mathbb{A}^2$ at the origin
2. the projection $\mathbb{P}^3 \supseteq Q = V(z_0z_3 - z_1z_2) \dashrightarrow \mathbb{P}^2$ from the point $p = [0,0,0,1]$ of sheet 6, exercise **2**, given by $[z_0, z_1, z_2, z_3] \mapsto [z_0, z_1, z_2]$
3. the standard quadratic transformation $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ of sheet 6, exercise **4** and **5**, given by $[z_0, z_1, z_2] \mapsto [z_1z_2, z_0z_2, z_1z_2]$,

see also sheet 7, exercise **6b**. To this end, let Δ be a fan in $(\mathbb{R}^2, \mathbb{Z}^2)$ such that X_Δ is smooth and let $\sigma \in \Delta$ be of dimension two.

a) Using sheet 7, exercise **5b**, show that there is a basis v_1, v_2 of $\mathbb{Z}^2$ such that $\sigma = \sigma(v_1, v_2)$.

b) Set $v := v_1 + v_2$. In Δ, we replace σ by the cones $\sigma(v_1, v)$ and $\sigma(v, v_2)$ producing a new fan Δ' in $(\mathbb{R}^2, \mathbb{Z}^2)$. Show that there is a natural $\mathbb{T}^2$-equivariant morphism $X_{\Delta'} \to X_\Delta$ and that it identifies with the blow-up morphism $\pi : \mathrm{Bl}_{x_\sigma} X_\Delta \to X_\Delta$ of X_Δ at x_σ. Here x_σ is the unique $\mathbb{T}^2$-fixed point in X_σ. It can be defined as in sheet 7, exercise **5b**, by

$$S_\sigma \to \mathbb{C}\ ,\ u \mapsto \begin{cases} 1 & u = 0 \\ 0 & u \neq 0 \end{cases} .$$

2. Go through the proof of Harris, Theorem 3.5, which shows that $\pi_p(X)$ is a projective variety using the resultant. Here $\pi_p : \mathbb{P}^n - \{p\} \to \mathbb{P}^{n-1}$ is the projection from a point $p \in \mathbb{P}^n$ and X is a projective variety in $\mathbb{P}^n$ not containing p.

3. Let us assume $\operatorname{char} k = 0$. We identify the space of quadrics in $\mathbb{P}^2$ with $\mathbb{P}^5$ in the natural way. Let $\Sigma_1 \subseteq \Sigma_2$ be the space of quadrics of rank one and of rank ≤ 2 respectively. Show

a) Σ_1 is the image of a Veronese embedding $\nu_2 : \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$.

b) Σ_2 is defined by a single cubic equation. The set of singular points in Σ_2 is Σ_1.

Hint: Jacobian criterion

4. Let us assume $\operatorname{char} k = 0$.

a) Describe the singular points of $V_f := V(y^2 - f(x)) \subseteq \mathbb{A}^2$ in terms of $f \in k[x]$. When is V_f smooth, when irreducible?

Hint: Jacobian criterion

b) Let $d \geq 1$. Under the identification of $\{f \in k[x] \mid f(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0\} \cong \mathbb{A}^d$ show that the sets

$$\{f \mid V_f \text{ smooth }\} \subseteq \{f \mid V_f \text{ irreducible }\} \subseteq \mathbb{A}^d$$

are both open. Compute the codimension of their complements in $\mathbb{A}^d$.

5. Let X be a variety. Show that $X \to \mathbb{Z}_{\geq 0}$, $p \mapsto \dim \mathrm{T}_p X$, is *upper semicontinuous*, i.e. $\{p \in X \mid \dim \mathrm{T}_p X \geq n\}$ is closed for any $n \in \mathbb{Z}_{\geq 0}$.

Due on Friday, May 8.