Sheet 9

Unless stated otherwise we work over an algebraically closed field k.

1. Let X be a variety. A subset of X is *locally closed* if it is the intersection of an open and a closed subset of X. Show

a) A subset of X is locally closed if and only if it is open in its closure.

b) A subset of X is constructible if and only if it is a finite disjoint union of locally closed subsets.

c) Every constructible subset of X contains an open dense subset of its closure.

2. Let G be an algebraic group acting on a variety X. Show

a) Each orbit Gx, $x \in X$, is a locally closed subset of X.

Hint: First show that Gx is constructible, then use **1c**.

b) The boundary $\overline{Gx} \setminus Gx$ of Gx is a union of orbits of strictly lower dimension. Thus, orbits of minimal dimension are closed.

c) If G is connected, then each orbit is an irreducible subset of X.

3. Let A be an algebraic group. A is called *abelian variety* if it is connected and proper. Show

a) Let X, Y, Z be varieties. Let X be proper and Y be irreducible. Let $\varphi : X \times Y \to Z$ be a morphism. If there is a $y_0 \in Y$ such that $\varphi(\cdot, y_0) : X \to Z$ is constant, then there is a morphism $\psi : Y \to Z$ such that $\varphi(\cdot, y) = \psi(y)$ for all $y \in Y$.

b) Let A and B be abelian varieties. Then, any morphism $\alpha : A \to B$ is the composite of a group homomorphism with a translation. In particular, if $\alpha(1_A) = 1_B$ then α is a group homomorphism.

c) If A is an abelian variety, then the group law is commutative.

Hint: Consider the inverse morphism $\iota : A \to A$, $a \mapsto a^{-1}$.

4. *(Valuative criterion for properness)* Let X be an irreducible variety. Assume that for all irreducible closed $Z \subseteq X$ and for all valuation rings R of $k(Z)$ containing k there is a $p \in Z$ such that $\mathcal{O}_{Z,p} \subseteq R$. Proceed as follows to show that X is proper.

a) Show that it suffices to prove: Given

1. Y an affine variety,
2. $V \subseteq Z \times Y$ irreducible closed,
3. $p_Y|V : V \to Y$ and $p_Z|V : V \to Z$ dominant, where $p_{Y,Z} : Z \times Y \to Y, Z$ denote the projections,

it follows that $p_Y|V$ surjects.

b) Prove the implication of **a** by applying the following lemma: Let A be an integral domain contained in a field $K \supseteq k$ and $\phi : A \to k$ be a homomorphism. Then there exists a valuation ring B of K containing A and a homomorphism $\Phi : B \to k$ such that $\Phi|A = \phi$.

5. Compute the integral closure of

a) $k[x, y]/(y^2 - x^3)$

b) $k[x, y]/(y^2 - x^2 - x^3)$

c) $k[x_0, x_1, \ldots, x_n]/(x_0^2 + x_1^2 + \cdots + x_r^2)$, $n \geq r \geq 2$, $\operatorname{char} k \neq 2$.

Due on Friday, May 15.