Problem Set 1

Ext groups

1. (a) Let A be an abelian group. Show that $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, A) \cong A/nA$.

   (b) Let A be a free abelian group and let B be another abelian group. Show that $\mathrm{Ext}(A, B) \cong 0$.

   (c) An abelian group A is called *divisible* if for any $a \in A$ and any nonzero $n \in \mathbb{Z}$ there exists an $a' \in A$ such that $a = na'$. A basic example is $\mathbb{Q}$ (with addition).

   Show that for any abelian group A and any divisible abelian group B the Ext group $\mathrm{Ext}(A, B)$ vanishes.

   Hint. *First show that any divisible group I is an injective object in the category of abelian groups, i.e. that for any exact sequence*

   $$0 \to B \to C$$

   of abelian groups and any homomorphism $f : B \to I$, there exists an extension $f' : C \to I$ such that the diagram

   $$\begin{tikzcd} & I & \\ 0 \arrow[r] & B \arrow[u, "f"] \arrow[r] & C \arrow[ul, dashed, "f'"'] \end{tikzcd}$$

   commutes.

   (d) For abelian groups A, A' and B, show that there is an isomorphism $\mathrm{Ext}(A \oplus A', B) \cong \mathrm{Ext}(A, B) \oplus \mathrm{Ext}(A', B)$.

2. (a) Compute $\mathrm{Ext}(\mathbb{Z}, \mathbb{Z})$, $\mathrm{Ext}(\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$, $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$ and $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$.

   (b) Compute $\mathrm{Ext}(\mathbb{Z}, \mathbb{Q})$, $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q})$, $\mathrm{Ext}(\mathbb{Q}, \mathbb{Q})$.

   (c) Compute $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z}/m\mathbb{Z})$.

3. Let A, A', B, C be abelian groups and

   $$0 \to A_1 \to A_0 \to A \to 0$$
   $$0 \to A_1' \to A_0' \to A' \to 0$$

   be free resolutions of A and A'.

   (a) For a homomorphism $f : B \to C$ define, the induced map

   $$\mathrm{Ext}(A, f) : \mathrm{Ext}(A, B) \to \mathrm{Ext}(A, C)$$

   by using the natural map

   $$\mathrm{Hom}(A, f) : \mathrm{Hom}(A_1, B) \to \mathrm{Hom}(A_1, C).$$

   Show that your definition is independent of the choices made.

   (b) For any homomorphism $f : A \to A'$, find homomorphisms to fill out the arrows of the following commutative diagram with exact rows.

   $$\begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r] \arrow[d, "\tilde{f}"'] \arrow[dr] & A_0 \arrow[r] \arrow[d, dashed] \arrow[dr] & A \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & A_1' \arrow[r] & A_0' \arrow[r] & A' \arrow[r] & 0 \end{tikzcd}$$

Using the homomorphism $\tilde{f}$ you have constructed, define the induced map

$$\mathrm{Ext}(f, B) : \mathrm{Ext}(A', B) \to \mathrm{Ext}(A, B).$$

Show that the definition is independent of the choices made.

(c) Show that the multiplication-by-n-maps $A \to A$, $B \to B$ induce the multiplication-by-n-map on $\mathrm{Ext}(A, B)$.

4. (*) We want to compute $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z})$.

(a) The *Prüfer group* $\mathbb{Z}(p^\infty)$ is defined as the quotient $\mathbb{Z}[p^{-1}]/\mathbb{Z}$ where $\mathbb{Z}[p^{-1}] \subset \mathbb{Q}$ is the ring of rational numbers whose denominator is a power of p.
Show that $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_p \mathbb{Z}(p^\infty)$.

(b) The ring $\mathbb{Z}_p$ of *p-adic integers* is the set of sequences $(a_n)_{n \geq 1}$ with $a_n \in \mathbb{Z}/p^n\mathbb{Z}$ such that for all $n \leq m$ we have that $a_n \equiv a_m \pmod{p^n}$, together with piecewise addition and multiplication. The field $\mathbb{Q}_p$ of *p-adic numbers* is the quotient field of $\mathbb{Z}_p$. For every element x of $\mathbb{Q}_p$ there exists an n such that $p^n x \in \mathbb{Z}_p$. Using the projections $\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ we can view $\mathbb{Z}$ as the subset of $\mathbb{Z}_p$ of constant sequences, and similarly $\mathbb{Q} \subset \mathbb{Q}_p$.
Prove that there exist isomorphisms

$$\mathrm{Hom}(\mathbb{Q}, \mathbb{Z}(p^\infty)) \cong \mathrm{Hom}(\mathbb{Z}[p^{-1}], \mathbb{Z}(p^\infty)) \cong \mathbb{Q}_p.$$

(c) In the lecture, it will be shown that the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

induces a long exact sequence

$$0 \to \mathrm{Hom}(\mathbb{Q}, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Q}, \mathbb{Q}) \to \mathrm{Hom}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Ext}(\mathbb{Q}, \mathbb{Z}) \to \mathrm{Ext}(\mathbb{Q}, \mathbb{Q}) \to \mathrm{Ext}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \to 0.$$

Use this fact and the previous parts show that $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z})$ can be described as the quotient $A/\mathbb{Q}$, where A is the *adèle group*, i.e. the subgroup of $\prod_p \mathbb{Q}_p$ of sequences $(x_2, x_3, x_5, \dots)$ such that all but finitely many of the x_p lie in $\mathbb{Z}_p$, and where we view $\mathbb{Q} \subset A$ as the subset of constant sequences.