Prof. Rahul Pandharipande
ETH Zürich

Problem Set 3

Cup products

1. (a) For $G \in \{\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}\}$, compute the ring structure for the cohomology with G-coefficients of the Klein bottle K.

   (b) Repeat the previous part for the connected sum of K with an oriented surface M of genus g, i.e. the space obtained by cutting out small disks from K and M, and glueing together the resulting boundaries along a degree one map.

2. (a) Using the cup product structure $H^k(X, A; R) \times H^l(X, B; R) \to H^{k+l}(X, A \cup B; R)$, show that if X is the union of contractible open subsets A and B, then all cup products of positive-dimensional classes in $H^*(X; R)$ are zero. What does this imply for the cohomology of the suspension of a space? Generalize to the situation that X is the union of n contractible open subsets, to show that all n-fold cup products of positive-dimensional classes are zero.

   (b) Show that for $X = \mathbb{C}P^n$ the minimum number of contractible open sets X can be covered with, is $n + 1$.

   (c) Show that for a closed surface X of genus $g \geq 1$ exactly 3 contractible open sets are necessary to cover X.

3. Let X be $\mathbb{C}P^2$ with a cell e^3 attached by a map $S^2 \to \mathbb{C}P^1 \subset \mathbb{C}P^2$ of degree p, and let $Y = M(\mathbb{Z}/p\mathbb{Z}, 2) \vee S^4$, where $M(\mathbb{Z}/p\mathbb{Z}, 2)$ is the Moore space as in problem 4 of problem set 2. Thus X and Y have the same 3-skeleton but differ in the way their 4-cells are attached. Show that X and Y have isomorphic cohomology rings with $\mathbb{Z}$ coefficients but not with $\mathbb{Z}/p\mathbb{Z}$ coefficients.

4. (a) Using the cup product structure, show there is no map $\mathbb{R}P^n \to \mathbb{R}P^m$ inducing a nontrivial map $H^1(\mathbb{R}P^m; \mathbb{Z}/2\mathbb{Z}) \to H^1(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$ if $n > m$. What is the corresponding result for maps $\mathbb{C}P^n \to \mathbb{C}P^m$?

   (b) Prove the Borsuk–Ulam theorem using part a).
   Hint. If $f : S^n \to \mathbb{R}^n$ is a map satisfying $f(x) \neq f(-x)$ for all x, it might be useful to consider the map $g : S^n \to S^{n-1}$ by $g(x) = (f(x) - f(-x))/|f(x) - f(-x)|$.

   (c) Use Borsuk-Ulam to prove the following statement: Let $S^n = \bigcup_i U_i$ be covering of S^n by $(n + 1)$ open sets U_i. Then there exists an i such that U_i contains a point x and its antipodal $-x$.