Problem Set 5

Cohomology with compact support, Poincaré duality

1. Show that direct limits commute with homology, i.e. that for a directed system of chain complexes $\{C_\alpha, f_{\alpha\beta}\}$, where $f_{\alpha\beta} : C_\alpha \to C_\beta$ are the chain maps, it holds $H_n(\varinjlim C_\alpha) = \varinjlim H_n(C_\alpha)$. In particular, direct limits preserve exact sequences.

2. Show that $H^0_c(X;G) = 0$ if X is path-connected and noncompact.

3. Show that $H^n_c(X \times \mathbb{R}^m;G) \cong H^{n-m}_c(X;G)$ for all $m \leq n$.

4. Show that after a suitable change of basis, a skew-symmetric nonsingular bilinear form over $\mathbb{Z}$ can be represented by a matrix consisting of 2×2 blocks $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ along the diagonal and zeros elsewhere.

5. Let $\pi : \tilde{M} \to M$ be the two-sheeted orientable cover of the nonorientable closed n-manifold M. Show that $H^k(\tilde{M};F) \cong H^k(M;F) \oplus H^{n-k}(M;F)$ for the coefficient field $F = \mathbb{Q}$ or $F = \mathbb{Z}_p$ with p an odd prime, by filling in details in the following outline:
   (a) For a vector space V and a linear endomorphism $T : V \to V$ such that $T^2 = \mathrm{Id}_V$ there is a splitting $V = V^+ \oplus V^-$ of V into eigenspaces of T for the eigenvalues $+1$ and -1.
   (b) Use the non-trivial Deck transformation $\tau : \tilde{M} \to \tilde{M}$ (interchanging the two sheets) to define a splitting $H_k(\tilde{M};F) = H^+_k(\tilde{M};F) \oplus H^-_k(\tilde{M};F)$ as in Part a. Do likewise in cohomology.
   (c) Using τ, define a natural isomorphism $H_k(M;F) \to H^+_k(\tilde{M};F)$ and proceed similarly in cohomology. Use these isomorphisms to identify the relevant vector spaces.
   (d) Show that the Poincaré duality isomorphism identifies the $+$ and $-$ parts of $H^k(\tilde{M};F)$ with the $-$ and $+$ parts of $H_{n-k}(\tilde{M};F)$, respectively, using the fact that $\tau_*[\tilde{M}] = -[\tilde{M}]$.

   Where did you need to use that $p \neq 2$?

6. For connected n-manifolds M_1 and M_2 the *connected sum* $M_1 \sharp M_2$ can be constructed by cutting out small n-dimensional balls from M_1 and M_2 and gluing the resulting boundary spheres via a homeomorphism. For example, the connected sum of a surface of genus g with a surface of genus h is a surface of genus $g + h$.
   (a) Determine under which conditions $M_1 \sharp M_2$ is orientable.
   (b) Compute $H_*(M_1 \sharp M_2)$ in terms of $H_*(M_1)$ and $H_*(M_2)$.
   Hint. One possible approach is to compare $M_1 \sharp M_2$ to the one-point union $M_1 \vee M_2$.
   (c) Compute the cup product structure in $H^*((S^2 \times S^8)\sharp(S^4 \times S^6);\mathbb{Z})$, and in particular show that the only nontrivial cup products are those dictated by Poincaré duality.