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Algebraic Topology II
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Problem Set 1

Ext groups

1. (a) Let A be an abelian group. Show that $\operatorname{Ext}(\mathbb{Z}/n\mathbb{Z}, A) \cong A/nA$.

   Solution: We use the free resolution $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ of $\mathbb{Z}/n\mathbb{Z}$. This induces the map $f(x) \mapsto f(nx) = nf(x)$ on $\operatorname{Hom}(\mathbb{Z}, A)$, i.e. multiplication by n. We identify $\operatorname{Hom}(\mathbb{Z}, A)$ with A by the image of $1 \in \mathbb{Z}$. Under this identification we see that by definition $\operatorname{Ext}(\mathbb{Z}/n\mathbb{Z}, A)$ is the cokernel of the multiplication-by-n-map on A. So $\operatorname{Ext}(\mathbb{Z}/n\mathbb{Z}, A) \cong A/nA$.

   (b) Let A be a free abelian group and let B be another abelian group. Show that $\operatorname{Ext}(A, B) \cong 0$.

   Solution: The sequence $0 \to 0 \to A \to A \to 0$ is a free resolution of A. Therefore $\operatorname{Ext}(A, B)$ is a quotient of $\operatorname{Hom}(0, B) \cong 0$ and therefore trivial.

   (c) An abelian group A is called *divisible* if for any $a \in A$ and any nonzero $n \in \mathbb{Z}$ there exists an $a' \in A$ such that $a = na'$. A basic example is $\mathbb{Q}$ (with addition).

   Show that for any abelian group A and any divisible abelian group B the Ext group $\operatorname{Ext}(A, B)$ vanishes.

   Hint. *First show that any divisible group I is an injective object in the category of abelian groups, i.e. that for any exact sequence*

   $$0 \to B \to C$$

   of abelian groups and any homomorphism $f : B \to I$, there exists an extension $f' : C \to I$ such that the diagram

   $$\begin{tikzcd}
 & I & \\
0 \arrow[r] & B \arrow[u, "f"] \arrow[r] & C \arrow[ul, dashed, "f'"']
\end{tikzcd}$$

   commutes.

   Solution: We first prove the lifting property as in the Hint. Let C' be a maximal subgroup of C such that a lift exists. Assume that $C' \neq C$ so that there is an $x \in C \setminus C'$. By divisibility of I also no integer multiple of x can lie in C'. But then we can extend the map to the group generated by C' and x by assigning to x an arbitrary element of I. This is in contradiction to minimality.

   Let now $0 \to A_1 \to A_0 \to A \to 0$ be a free resolution of A. Since B is an injective object and $A_1 \to A_0$ is injective we have that $\operatorname{Hom}(A_0, B) \to \operatorname{Hom}(A_1, B)$ is surjective and therefore the Ext group $\operatorname{Ext}(A, B)$ vanishes.

   (d) For abelian groups A, A' and B, show that there is an isomorphism $\operatorname{Ext}(A \oplus A', B) \cong \operatorname{Ext}(A, B) \oplus \operatorname{Ext}(A', B)$.

   Solution: If $0 \to A_1 \to A_0 \to A \to 0$ and $0 \to A_1' \to A_0' \to A' \to 0$ are free resolutions of A and A' respectively, $0 \to A_1 \oplus A_1' \to A_0 \oplus A_0' \to A \oplus A' \to 0$ (with diagonal maps) is a free resolution of $A \oplus A'$. The necessary isomorphism follows easily from the isomorphisms $\operatorname{Hom}(A_1 \oplus A_1', B) \cong \operatorname{Hom}(A_1, B) \oplus \operatorname{Hom}(A_1', B)$ and $\operatorname{Hom}(A_0 \oplus A_0', B) \cong \operatorname{Hom}(A_0, B) \oplus \operatorname{Hom}(A_0', B)$.

2. (a) Compute $\operatorname{Ext}(\mathbb{Z}, \mathbb{Z})$, $\operatorname{Ext}(\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$, $\operatorname{Ext}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$ and $\operatorname{Ext}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$.

   Solution: By Exercise 1b $\operatorname{Ext}(\mathbb{Z}, \mathbb{Z}) \cong \operatorname{Ext}(\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong 0$. By 1a $\operatorname{Ext}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$. Also by 1a $\operatorname{Ext}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$, where d is the greatest common divisor of n and m.

   (b) Compute $\operatorname{Ext}(\mathbb{Z}, \mathbb{Q})$, $\operatorname{Ext}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q})$, $\operatorname{Ext}(\mathbb{Q}, \mathbb{Q})$.

Solution: By Exercise 1c all groups vanish.

(c) Compute $\operatorname{Ext}(\mathbb{Q}, \mathbb{Z}/m\mathbb{Z})$.

Solution: Let $0 \to Q_1 \to Q_0 \to \mathbb{Q} \to 0$ be a free resolution of $\mathbb{Q}$. The mth multiple of any element in $\operatorname{Ext}(\mathbb{Q}, \mathbb{Z}/m\mathbb{Z}) \cong \operatorname{coker}(\operatorname{Hom}(Q_0, \mathbb{Z}/m\mathbb{Z}) \to \operatorname{Hom}(Q_1, \mathbb{Z}/m\mathbb{Z}))$ is zero. On the other hand $\operatorname{Ext}(\mathbb{Q}, \mathbb{Z}/m\mathbb{Z})$ is also a $\mathbb{Q}$-vector space, where the scalar multiplication with a rational number is defined by the induced map on Ext groups as in Exercise 3c. These two properties of $\operatorname{Ext}(\mathbb{Q}, \mathbb{Z}/m\mathbb{Z})$ imply that $\operatorname{Ext}(\mathbb{Q}, \mathbb{Z}/m\mathbb{Z}) \cong 0$.

3. Let A, A', B, C be abelian groups and

$$0 \to A_1 \to A_0 \to A \to 0$$
$$0 \to A'_1 \to A'_0 \to A' \to 0$$

be free resolutions of A and A'.

(a) For a homomorphism $f : B \to C$ define, the induced map

$$\operatorname{Ext}(A, f) : \operatorname{Ext}(A, B) \to \operatorname{Ext}(A, C)$$

by using the natural map

$$\operatorname{Hom}(A, f) : \operatorname{Hom}(A_1, B) \to \operatorname{Hom}(A_1, C).$$

Show that your definition is independent of the choices made.

Solution: We apply the homomorphism theorem. In order to show that $\operatorname{Hom}(A_1, f)$ induces a map between the Ext groups we need to show that $\operatorname{Hom}(A_1, f)$ sends maps which are compositions of the map $A_1 \to A_0$ with elements in $\operatorname{Hom}(A_0, B)$ to maps which are compositions of $A_1 \to A_0$ with elements from $\operatorname{Hom}(A_0, C)$, but this is obvious from the definitions.

(b) For any homomorphism $f : A \to A'$, find homomorphisms to fill out the arrows of the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r] \arrow[d, "\tilde{f}"'] \arrow[dr] & A_0 \arrow[r] \arrow[d, dashed] \arrow[dr] & A \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & A'_1 \arrow[r] & A'_0 \arrow[r] & A' \arrow[r] & 0
\end{tikzcd}$$

Using the homomorphism $\tilde{f}$ you have constructed, define the induced map

$$\operatorname{Ext}(f, B) : \operatorname{Ext}(A', B) \to \operatorname{Ext}(A, B).$$

Show that the definition is independent of the choices made.

Solution: From $f : A \to A'$ we obtain a map $A_0 \to A'$ by composition with $A_0 \to A$. Since A_0 is free we can find a lift $f_0 : A_0 \to A'_0$. By composition with $A_1 \to A_0$ we obtain a map $A_1 \to A'_0$. By exactness and commutativity of the diagram the composition $A_1 \to A'_0 \to A'$ is zero and the image of $A_1 \to A'_0$ in A'_0 is contained in the image of A'_1. This induces a map $\tilde{f} : A_1 \to A'_1$. This map depends in general on the choice of the lift f_0.

Similarly to Exercise 3a, the map $\tilde{f}$ induces a map $\operatorname{Ext}(f, B)$ between the Ext groups.

We want to show that this map does not depend on the choice of lift f_0. Without loss of generality we can assume that $f = 0$. Since the composition of f_0 with $A'_0 \to A'$ is zero and by exactness of the second row, f_0 is the composition of a map $A_0 \to A'_1$ and therefore by commutativity $\tilde{f}$ is the composition of $A_1 \to A_0$ and this map. This however implies that the map $\operatorname{Hom}(\tilde{f}, B)$ factors through a map $\operatorname{Hom}(A'_1, B) \to \operatorname{Hom}(A_0, B)$ and therefore vanishes when we pass to the quotient.

(c) Show that the multiplication-by-n-maps $A \to A$, $B \to B$ induce the multiplication-by-n-map on $\mathrm{Ext}(A, B)$.

Solution: Let $0 \to A_1 \to A_0 \to A \to 0$ be a free resolution of A. Obviously the multiplication-by-n-map on B induces the multiplication by map on $\mathrm{Hom}(A_1, B)$ and therefore also on $\mathrm{Ext}(A, B)$. The multiplication-by-n-map on A can be lifted in the diagram of Exercise 3b to the multiplication-by-n-map on A_1. By the observation in Exercise 1a this induces the multiplication-by-n-map on $\mathrm{Hom}(A_1, B)$ and therefore also on $\mathrm{Ext}(A, B)$.

4. (*) We want to compute $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z})$.

(a) The *Prüfer group* $\mathbb{Z}(p^\infty)$ is defined as the quotient $\mathbb{Z}[p^{-1}]/\mathbb{Z}$ where $\mathbb{Z}[p^{-1}] \subset \mathbb{Q}$ is the ring of rational numbers whose denominator is a power of p.

Show that $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_p \mathbb{Z}(p^\infty)$.

Solution: This is an easy consequence of partial fraction decomposition.

(b) The ring $\mathbb{Z}_p$ of *p-adic integers* is the set of sequences $(a_n)_{n\geq 1}$ with $a_n \in \mathbb{Z}/p^n\mathbb{Z}$ such that for all $n \leq m$ we have that $a_n \equiv a_m \pmod{p^n}$, together with piecewise addition and multiplication. The field $\mathbb{Q}_p$ of *p-adic numbers* is the quotient field of $\mathbb{Z}_p$. For every element x of $\mathbb{Q}_p$ there exists an n such that $p^n x \in \mathbb{Z}_p$. Using the projections $\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ we can view $\mathbb{Z}$ as the subset of $\mathbb{Z}_p$ of constant sequences, and similarly $\mathbb{Q} \subset \mathbb{Q}_p$.

Prove that there exist isomorphisms

$$\mathrm{Hom}(\mathbb{Q}, \mathbb{Z}(p^\infty)) \cong \mathrm{Hom}(\mathbb{Z}[p^{-1}], \mathbb{Z}(p^\infty)) \cong \mathbb{Q}_p.$$

Solution: The first isomorphism follows from the decomposition of 4a and the fact that for different primes p, q the group $\mathrm{Hom}(\mathbb{Z}(p^\infty), \mathbb{Z}(q^\infty))$ is trivial since for a homomorphism f in this group there would need to exponents e, e' such that $p^e f = q^{e'} f = 0$ which implies (since p^e and $q^{e'}$ are coprime) that $f = 0$.

We construct a map $m : \mathbb{Z}_p \to \mathrm{Hom}(\mathbb{Z}[p^{-1}], \mathbb{Z}(p^\infty))$ by assigning to a sequence $(a_n) \in \mathbb{Z}_p$, the map which maps $\frac{b}{p^n}$ to $\frac{a_n b}{p^n}$ modulo $\mathbb{Z}$. The map m is well-defined since $\frac{bp^k}{p^{n+k}}$ is mapped to $\frac{a_{n+k}b}{p^n}$, which is also $\frac{a_n b}{p^n}$ modulo $\mathbb{Z}$. It is easy to see that m is a group homomorphism.

There is a unique extension of m to a map $m : \mathbb{Q}_p \to \mathrm{Hom}(\mathbb{Z}[p^{-1}], \mathbb{Z}(p^\infty))$ defined by $m(ap^{-k}) = m(a)p^{-k}$ for $a \in \mathbb{Z}_p$.

To check that m is an isomorphism, we show that it is both injective and surjective. If there is a nonzero element a in the kernel of m, we can assume without loss of generality that it lies in $\mathbb{Z}_p$ and by definition this implies that $a_n \equiv 0$ modulo p^n and therefore that $a = 0$.

For surjectivity we construct the preimage of any homomorphism $f : \mathbb{Z}[p^{-1}] \to \mathbb{Z}(p^\infty)$. The homomorphism is uniquely determined from the images b_n of p^{-n} and these need to satisfy $pb_{n+1} \equiv b_n$ modulo $\mathbb{Z}$. Therefore there exists a k such that $p^{-1}a_n := p^{k-1+n}b_n \in p^{-1}\mathbb{Z}/\mathbb{Z} \cong \mathbb{Z}/p\mathbb{Z}$. These a_n have to satisfy $a_n \equiv a_{n+1} \pmod{p^n}$ and therefore give an element $a \in \mathbb{Z}_p$. It is now easy to see that $ap^{-k} \in \mathbb{Q}_p$ is a preimage of m.

(c) In the lecture, it will be shown that the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

induces a long exact sequence

$$0 \to \mathrm{Hom}(\mathbb{Q}, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Q}, \mathbb{Q}) \to \mathrm{Hom}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Ext}(\mathbb{Q}, \mathbb{Z}) \to \mathrm{Ext}(\mathbb{Q}, \mathbb{Q}) \to \mathrm{Ext}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \to 0.$$

Use this fact and the previous parts show that $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z})$ can be described as the quotient $A/\mathbb{Q}$, where A is the *adèle group*, i.e. the subgroup of $\prod_p \mathbb{Q}_p$ of sequences $(x_2, x_3, x_5, \dots)$ such that all but finitely many of the x_p lie in $\mathbb{Z}_p$, and where we view $\mathbb{Q} \subset A$ as the subset of constant sequences.

Solution: From Exercise 1c we know that the last two groups in the long exact sequence vanish. Therefore $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z})$ is the cokernel of the map $\mathrm{Hom}(\mathbb{Q}, \mathbb{Q}) \to \mathrm{Hom}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z})$. The group $\mathrm{Hom}(\mathbb{Q}, \mathbb{Q})$ is isomorphic to $\mathbb{Q}$ and by Exercises 4a and 4b there are isomorphisms

$$\mathrm{Hom}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Q}, \bigoplus_p \mathbb{Z}(p^\infty)) \subset \mathrm{Hom}(\mathbb{Q}, \prod_p \mathbb{Z}(p^\infty)) \cong \prod_p \mathbb{Q}_p.$$

We check that the subset $\mathrm{Hom}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z})$ gives inside $\prod_p \mathbb{Q}_p$ is isomorphic to the ring A of adèles by comparing the definition of A to the definition of the direct sum $\bigoplus_p \mathbb{Z}(p^\infty)$. Finally we check that $\mathbb{Q}$ is embedded in A as described in the exercise.