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Algebraic Topology II
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Problem Set 3

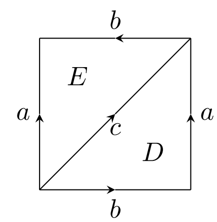
Cup products

1. (a) For $G \in \{\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}\}$, compute the ring structure for the cohomology with G-coefficients of the Klein bottle K.

   Solution: Since the Klein bottle is connected and has only nontrivial H^0, H^1 and H^2, the ring structure is determined from the cup product $H^1 \times H^1 \to H^2$.

   Recall that $H^1(K; \mathbb{Z}) \cong \mathbb{Z}$ and $H^2(K; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Therefore for $G = \mathbb{Z}$ this cup product map has to be zero.

   For $G = \mathbb{Z}/2\mathbb{Z}$ we have $H^1(K; G) \cong (\mathbb{Z}/2\mathbb{Z})^2$ and $H^2(K; G) \cong \mathbb{Z}/2\mathbb{Z}$ and there can be nontrivial cup products. Consider the following simplicial complex.



   We use the notation a^* to denote the cochain dual to a. Then $a^* + c^*$ and $b^* + c^*$ give a basis of the first cohomology. We want to calculate their three possible cup products.

   We find that out of the 9 possible cup products of a^*, b^* and c^* only

$$b^* \cup a^* = D^*, \quad c^* \cup b^* = E^*$$

   are nonzero. Notice that $D^* = E^*$ in cohomology. So we have that $(a^* + c^*)^2 = 0$, $(a^* + c^*) \cup (b^* + c^*) = D^*$ and $(b^* + c^*)^2 = D^*$.

   In total we can write $H^*(K, \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})[x, y]/(y^2, x^3, x^2 - xy)$ as rings.

   (b) Repeat the previous part for the connected sum of K with an oriented surface M of genus g, i.e. the space obtained by cutting out small disks from K and M, and glueing together the resulting boundaries along a degree one map.

   Solution: Let X be the above described topological space. It can be obtained from a $(4g+4)$-gon by glueing the first 4 edges as for the Klein bottle and the last $4g$ as for the genus g surface. The resulting CW-structure has one 0-cell, $(2g + 2)$ 1-cells and one 2-cell. For the resulting cellular cochain complex the only interesting coboundary map, from 1-cochains to 2-cochains, sends all 1-cochains to zero, but one which is sent to twice the generator of H^2. So the $\mathbb{Z}$-valued cohomolgy is $H^0(X; \mathbb{Z}) \cong \mathbb{Z}$, $H^1(X; \mathbb{Z}) \cong \mathbb{Z}^{2g+1}$, $H^2(X; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ whereas $H^0(X; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, $H^1(X; \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^{2g+2}$, $H^2(X; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

   Again $H^1(X; \mathbb{Z}) \times H^1(X; \mathbb{Z})$ has to be trivial.

   Consider the map $f : X \to K \vee M$ which contracts the S^1 that K and M are glued along together. The induced map on cohomology is an isomorphism on H^1 and sends each of the generators of $H^2(K; \mathbb{Z}/2\mathbb{Z})$ and $H^2(M; \mathbb{Z}/2\mathbb{Z})$ to the generator of $H^2(X; \mathbb{Z}/2\mathbb{Z})$. So we can read off all cup products between classes in $H^1(X; \mathbb{Z}/2\mathbb{Z})$ from cup products of K and M. Therefore the cohomology ring of X with $(\mathbb{Z}/2\mathbb{Z})$-coefficients is the direct product of the cohomology rings of K and M with H^0 and H^2 identified.

2. (a) Using the cup product structure $H^k(X, A; R) \times H^l(X, B; R) \to H^{k+l}(X, A \cup B; R)$, show that if X is the union of contractible open subsets A and B, then all cup products of positive-dimensional classes in $H^*(X; R)$ are zero. What does this imply for the cohomology of the suspension of a

space? Generalize to the situation that X is the union of n contractible open subsets, to show that all n-fold cup products of positive-dimensional classes are zero.

Solution: Since A and B are contractible $H^i(A;R) = H^i(B;R) = 0$ for $i \geq 1$. Therefore the long exact sequence implies that $H^i(X;R) \cong H^i(X,A;R) \cong H^i(X,B;R)$ for $i \geq 1$. Also, since $A \cup B = X$ we have that $H^i(X, A \cup B; R) = 0$ for $i \geq 1$. By definition of the relative cup product the diagram

$$\begin{tikzcd} H^k(X,A;R) \times H^l(X,B;R) \arrow[r] \arrow[d] & H^{k+l}(X, A \cup B; R) \arrow[d] \\ H^k(X;R) \times H^l(X;R) \arrow[r] & H^{k+l}(X;R) \end{tikzcd}$$

commutes. Since for $i \geq 1$ the left arrow is an isomorphism we see that all cup products of positive-dimensional classes in $H^*(X;R)$ are zero.

The suspension SX of X can be covered by two open sets which are homeomorphic to $X' = (X \times [0,1])/\sim$ where $\sim$ identifies all points whose project to the second factor is 1. Letting the second factor approach 1, we see that X' is contractible. Therefore for SX all cup products of positive-dimensional classes are vanish.

If X is a union $X = \bigcup_{i=1}^n A_i$ of contractible open sets A_i, we can compose the relative cup product maps to obtain for every $k = \sum_{i=1}^n k_i$ a commutative diagram

$$\begin{tikzcd} \prod_{i=1}^n H^{k_i}(X, A_i; R) \arrow[r] \arrow[d] & H^k(X, \bigcup_{i=1}^n A_i; R) \arrow[d] \\ \prod_{i=1}^n H^{k_i}(X;R) \arrow[r] & H^k(X;R) \end{tikzcd}.$$

By the same argument as before, the left arrow is an isomorphism if all $k_i \geq 1$. Therefore all cup products of n positive-dimensional classes are zero.

(b) Show that for $X = \mathbb{C}P^n$ the minimum number of contractible open sets X can be covered with, is $n+1$.

Solution: There is a well-known covering of X by $n+1$ charts. The n-fold cup product power of a generator of H^2 is nontrivial. Therefore it is not possible to cover X with n contractible open sets.

(c) Show that for a closed surface X of genus $g \geq 1$ exactly 3 contractible open sets are necessary to cover X.

Solution: For X the cup product map $H^1 \times H^1 \to H^2$ is nontrivial. Therefore at least 3 contractible open sets are necessary to cover X.

To construct such a covering we view X as being glued together from a regular $4g$-gon Y and cover Y by open sets which are still contractible after applying the quotient map $Y \to X$. For small ε, let U_1 be a closed ε-neighborhood of ∇Y, U_2 an open 2ε-neighborhood of ∇Y, U_3 a closed 4ε-neighborhood of the corners of Y and U_4 be an open 5ε-neighborhood of the corners of Y. Then the covering of Y by $Y \setminus U_1$, $U_2 \setminus U_3$ and U_4 induces a suitable covering of X by 3 open, contractible sets.

3. Let X be $\mathbb{C}P^2$ with a cell e^3 attached by a map $S^2 \to \mathbb{C}P^1 \subset \mathbb{C}P^2$ of degree p, and let $Y = M(\mathbb{Z}/p\mathbb{Z}, 2) \vee S^4$, where $M(\mathbb{Z}/p\mathbb{Z}, 2)$ is the Moore space as in problem 4 of problem set 2. Thus X and Y have the same 3-skeleton but differ in the way their 4-cells are attached. Show that X and Y have isomorphic cohomology rings with $\mathbb{Z}$ coefficients but not with $\mathbb{Z}/p\mathbb{Z}$ coefficients.

Solution: X and Y have the same CW-complex structure but for X the 4-cell is attached to the 2-cell, whereas for Y the 4-cell is attached to the 0-cell. Therefore their cellular chain and cochain complexes agree and therefore also their cohomology groups $H^0 \cong G$, $H^2 \cong \ker(G \xrightarrow{\cdot p} G)$, $H^3 \cong G/pG$ and $H^4 \cong G$ with G coefficients.

For $G = \mathbb{Z}$ their second cohomology groups vanish and their ring structure has to agree for dimension reasons.

For $G = \mathbb{Z}/p\mathbb{Z}$ there is a difference in the cup product $H^2 \times H^2 \to H^4$. Here H^2 and H^4 are both isomorphic to $\mathbb{Z}/p\mathbb{Z}$. For Y the cup product has to be trivial since H^2 is pulled-back from $M(\mathbb{Z}/p\mathbb{Z}, 2)$ via the natural inclusion $M(\mathbb{Z}/p\mathbb{Z}, 2) \to Y$ but $M(\mathbb{Z}/p\mathbb{Z}, 2)$ has no H^4. For X the maps on H^2 and H^4 induced from $\mathbb{C}P^2 \to X$ are isomorphisms and therefore the cup product $H^2(X; \mathbb{Z}/p\mathbb{Z}) \times H^2(X; \mathbb{Z}/p\mathbb{Z}) \to H^4(X; \mathbb{Z}/p\mathbb{Z})$ is the same (nontrivial) as for $\mathbb{C}P^2$.

4. (a) Using the cup product structure, show there is no map $\mathbb{R}P^n \to \mathbb{R}P^m$ inducing a nontrivial map $H^1(\mathbb{R}P^m; \mathbb{Z}/2\mathbb{Z}) \to H^1(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$ if $n > m$. What is the corresponding result for maps $\mathbb{C}P^n \to \mathbb{C}P^m$?

   Solution: Recall that as a ring $H^*(\mathbb{R}P^m; \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})[x]/x^{m+1}$ where x has degree 1. The first result follows from the fact that for the image y of x in $H^1(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$ to be nontrivial we would need to have $y^{m+1} = 0$ but that implies $n \le M$.

   The same argument shows that there is no map $\mathbb{C}P^n \to \mathbb{C}P^m$ inducing a nontrivial map $H^2(\mathbb{C}P^m; \mathbb{Z}) \to H^2(\mathbb{C}P^n)$ if $n > m$.

   (b) Prove the Borsuk–Ulam theorem using part a).

   Hint. If $f : S^n \to \mathbb{R}^n$ is a map satisfying $f(x) \neq f(-x)$ for all x, it might be useful to consider the map $g : S^n \to S^{n-1}$ by $g(x) = (f(x) - f(-x))/|f(x) - f(-x)|$.

   Solution: We use a proof by contradiction as suggested by the hint. By definition, the map g from the hint satisfies $g(-x) = -g(x)$. Therefore it induces a map $\tilde{g} : \mathbb{R}P^n \to \mathbb{R}P^{n-1}$. By Part a) the induced map on H^1 with $\mathbb{Z}/2\mathbb{Z}$ coefficients is trivial.

   On the other hand, for $n \geq 2$ the map g on the covering spaces S^n and S^{n-1} sends the lift of a nontrivial path in $\pi_1(\mathbb{R}P^n)$ to a nontrivial element of $\pi_1(\mathbb{R}P^{n-1})$. (A nontrivial path has opposite start and end points.) Since the map between the commutative π_1 is non-trivial, so has to be the map in homology. Using the universal coefficient theorem it follows that the map on H^1 with $\mathbb{Z}/2\mathbb{Z}$ coefficients has to be nontrivial. Contradiction.

   For $n = 1$ a map $g : S^1 \to S^0$ satisfying $g(-x) = -g(x)$ obviously cannot exist because of connectedness of S^1.

   (c) Use Borsuk-Ulam to prove the following statement: Let $S^n = \bigcup_i U_i$ be covering of S^n by $(n+1)$ open sets U_i. Then there exists an i such that U_i contains a point x and its antipodal $-x$.

   Solution: Let $V_i \subset U_i$ be slightly smaller and closed, so that $S^n = \bigcup_i V_i$ is a covering of S^n by $(n+1)$ closed sets. Since S^n is perfectly normal, there exist continuous functions $f_i : S^n \to [0, 1]$ which are equal to 1 on V_i, 0 on $S^n \setminus U_i$ and strictly between 0 and 1 otherwise. Consider the map $f : S^n \to \mathbb{R}^n$ given by $(f_1, \ldots, f_n)$. By Borsuk-Ulam there is exists an $x \in S^n$ such that $f(x) = f(-x)$. If $x \in U_i$ for some $i \neq n+1$, then $f_i(x) = f_i(-x)$ implies that U_i contains both x and $-x$. On the other hand, if $x \in U_{n+1}$, we must also have $-x \in U_{n+1}$ because otherwise there would exist an $i \neq n+1$ such that $-x \in U_i$ and $1 = f_i(-x) = f_i(x)$. Since the U_i cover S^n we are done.