Problem Set 5

Cohomology with compact support, Poincaré duality

1. Show that direct limits commute with homology, i.e. that for a directed system of chain complexes $\{C_\alpha, f_{\alpha\beta}\}$, where $f_{\alpha\beta} : C_\alpha \to C_\beta$ are the chain maps, it holds $H_n(\varinjlim C_\alpha) = \varinjlim H_n(C_\alpha)$. In particular, direct limits preserve exact sequences.

   Solution: First, recall that since $f_{\alpha\beta}$ are chain maps, they preserve the subcomplexes of cycles and boundaries. We therefore can define a map $\varinjlim H_n(C_\alpha) \to H_n(\varinjlim C_\alpha)$ by choosing for an element of the left injective limit an representative cycle in C_α for some α, viewing it as a cycle in the injective limit and taking its homology class. Had we chosen another representative C'_β by definition of the injective limit we would still get the same image in homology.

   We can define an inverse map $H_n(\varinjlim C_\alpha) \to \varinjlim H_n(C_\alpha)$ by lifting the given homology class to a cycle of the limit, which can be represented by a cycle in C_α for some α and considering its image in homology as an element of the injective limit.

2. Show that $H^0_c(X; G) = 0$ if X is path-connected and noncompact.

   Solution: For any singular 0-cochain φ with compact support, there exists a compact set $K \subset X$ such that φ is zero on all points of $X \setminus K$. By assumption on X there is a point $x \in X \setminus K$ and for every $y \in K$ there is a path γ_y from x to y. If φ is a cocycle, we must have

   $$0 = \delta\varphi([\gamma_y]) = \varphi(y) - \varphi(x) = \varphi(y).$$

   Since φ takes the value zero at every point of X, it must be zero itself. So $H^0_c(X; G)$ vanishes.

3. Show that $H^n_c(X \times \mathbb{R}^m; G) \cong H^{n-m}_c(X; G)$ for all $m \leq n$.

   Solution: To ease notation we leave out the G in the following. All cohomology groups should be understood as with G-coefficients.

   By induction, it is enough to prove the statement in the case $m = 1$.

   In order to compute $H^n_c(X \times \mathbb{R})$ as a limit of the directed set of compact subsets of $X \times \mathbb{R}$ it is enough to restrict oneself to the compact subsets of the form $K \times I$ where $K \subset X$ is compact and $I \subset \mathbb{R}$ is a closed interval.

   We divide $\mathbb{R} \setminus I$ into its connected components $\mathbb{R} \setminus I = I^+ \cup I^-$. Then we can cover $X \times \mathbb{R} \setminus K \times I$ by A^+ and A^-, where

   $$A^\pm = X \times I^\pm \cup (X \setminus K) \times \mathbb{R}.$$

   Their intersection is given by

   $$A^+ \cap A^- = (X \setminus K) \times \mathbb{R}.$$

   Let us consider the Mayer-Vietoris sequence

   $$\to H^{n-1}(X \times \mathbb{R}, (X \times \mathbb{R}) \setminus (K \times I)) \to H^n(X \times \mathbb{R}, (X \setminus K) \times \mathbb{R}) \to H^n(X \times \mathbb{R}, A^+) \oplus H^n(X \times \mathbb{R}, A^-) \to .$$

   Since $(X \times \mathbb{R}, (X \setminus K) \times \mathbb{R}) \sim (X, X \setminus K)$ and $A^+ \sim A^- \sim X \times \mathbb{R}$, we see that $H^{n-1}(X \times \mathbb{R}, (X \times \mathbb{R}) \setminus (K \times I)) \cong H^n(X, X \setminus K)$. Taking the limit we obtain the desired result.

4. Show that after a suitable change of basis, a skew-symmetric nonsingular bilinear form over $\mathbb{Z}$ can be represented by a matrix consisting of 2×2 blocks $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ along the diagonal and zeros elsewhere.

Solution: Let V be a finitely generated $\mathbb{Z}$-module and $\langle , \rangle$ a nonsingular bilinear form. Choose a nonzero element x of V such that there does not exist an integer $n \geq 2$ and an element $y \in V$ such that $ny = x$. We can then define a linear form φ on V such that $\varphi(x) = 1$. By nonsingularity there must exist a $y \in V$ such that for all $z \in V$ we have $\varphi(z) = \langle z, y \rangle$. In particular, $\langle x, y \rangle = 1$. Because of the skew-symmetry $\langle , \rangle$ is described by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in the submodule generated by x and y.

Consider V_1, the submodule generated by x and y, and V_2, its orthogonal complement with respect to $\langle , \rangle$. In order to show that $V \cong V_1 \oplus V_2$, we construct a projection $p : V \to V_1$ by setting that

$$p(z) = \langle z, y \rangle x + \langle x, z \rangle y.$$

By construction, $\langle , \rangle$ restricts to the orthogonal submodules V_1 and V_2 and is nonsingular on each of them. We can conclude by induction (we reduce the rank or torsion in each step).

5. Let $\pi : \tilde{M} \to M$ be the two-sheeted orientable cover of the nonorientable closed n-manifold M. Show that $H^k(\tilde{M}; F) \cong H^k(M; F) \oplus H^{n-k}(M; F)$ for the coefficient field $F = \mathbb{Q}$ or $F = \mathbb{Z}_p$ with p an odd prime, by filling in details in the following outline:

   (a) For a vector space V and a linear endomorphism $T : V \to V$ such that $T^2 = \mathrm{Id}_V$ there is a splitting $V = V^+ \oplus V^-$ of V into eigenspaces of T for the eigenvalues $+1$ and -1.

   Solution: Since p is odd, there are projections p_+ and p_- to the eigenspaces given by $p_+(x) = \frac{1}{2}(x + Tx)$ and $p_-(x) = \frac{1}{2}(x - Tx)$.

   (b) Use the non-trivial Deck transformation $\tau : \tilde{M} \to \tilde{M}$ (interchanging the two sheets) to define a splitting $H_k(\tilde{M}; F) = H_k^+(\tilde{M}; F) \oplus H_k^-(\tilde{M}; F)$ as in Part a. Do likewise in cohomology.

   Solution: τ induces an endomorphism τ_* in homology and and an endomorphism τ^* in cohomology. Because τ^2 is the identity τ_*^2 and $(\tau^*)^2$ are the identity in homology and cohomology, respectively. By Part a, we obtain the desired splitting.

   (c) Using τ, define a natural isomorphism $H_k(M; F) \to H_k^+(\tilde{M}; F)$ and proceed similarly in cohomology. Use these isomorphisms to identify the relevant vector spaces.

   Solution: Using barycentric subdivision we can represent every element of $H_k(M; F)$ by a cycle such that each simplex lies in an open set of U such that $\pi^{-1}(U)$ is a disjoint union of two copies of U. We define a map $H_k(M; F) \to H_k^+(\tilde{M}; F)$ by saying that each simplex in such an open set U should be sent two the sum of the two preimages under π. The condition that τ_* acts by $+1$ on H_k^+ implies that the map is bijective. Because we are working over a field we can use the universal coefficient theorem to obtain the Hom-dual isomorphism in cohomology. Since we have not used anything special about M the map is natural.

   (d) Show that the Poincaré duality isomorphism identifies the $+$ and $-$ parts of $H^k(\tilde{M}; F)$ with the $-$ and $+$ parts of $H_{n-k}(\tilde{M}; F)$, respectively, using the fact that $\tau_*[\tilde{M}] = -[\tilde{M}]$.

   Solution: Using $\tau_*[\tilde{M}] = -[\tilde{M}]$ and functoriality of the cap product it is straightforward to check that the diagram

$$\begin{tikzcd} H^k(\tilde{M}; F) \arrow[r, "{[\tilde{M}]\cap}"] & H_{n-k}(\tilde{M}; F) \\ H^k(\tilde{M}; F)^+ \arrow[u] \arrow[r, "{[\tilde{M}]\cap}"] & H_{n-k}(\tilde{M}; F)^- \arrow[u] \end{tikzcd}$$

   commutes, e.g. $\varphi \in H^k(\tilde{M}; F)^+$ is mapped to $[\tilde{M}] \cap \varphi$, which satisfies

$$\tau_*([\tilde{M}] \cap \varphi) = \tau_*([\tilde{M}] \cap \tau^*\varphi) = \tau_*([\tilde{M}]) \cap \varphi = -[\tilde{M}] \cap \varphi.$$

The lower map is an injective map between equal dimensional vector space and hence and isomorphism. We can proceed similarly when $+$ and $-$ are reversed.

Where did you need to use that $p \neq 2$?

6. For connected n-manifolds M_1 and M_2 the *connected sum* $M_1 \sharp M_2$ can be constructed by cutting out small n-dimensional balls from M_1 and M_2 and gluing the resulting boundary spheres via a homeomorphism. For example, the connected sum of a surface of genus g with a surface of genus h is a surface of genus $g+h$.

   (a) Determine under which conditions $M_1 \sharp M_2$ is orientable.

   Solution: $M_1 \sharp M_2$ is glued from open sets which are isomorphic to M_1 and M_2 with a small disk removed and a cylinder $S^{n-1} \times]0,1[$. The orientation double cover of $M_1 \sharp M_2$ can be constructed by gluing the orientation double covers of the three pieces. Since the orientation double cover of the cylinder consists of two diconnected cylinders, the orientation double cover of $M_1 \sharp M_2$ will only be disconnected in the case that both the orientation double covers of M_1 and M_2 are disconnected. Hence, $M_1 \sharp M_2$ is orientable iff M_1 and M_2 are orientable.

   (b) Compute $H_*(M_1 \sharp M_2)$ in terms of $H_*(M_1)$ and $H_*(M_2)$.
   Hint. One possible approach is to compare $M_1 \sharp M_2$ to the one-point union $M_1 \vee M_2$.

   Solution: We consider the long exact sequence of the pair $(M_1 \sharp M_2, S^{n-1})$, where S^{n-1} is the boundary sphere used to glue M_1 and M_2 together. Since S^{n-1} is a neighborhood retract,

$$H_*(M_1 \sharp M_2, S^{n-1}) \cong \tilde{H}_*(M_1 \vee M_2).$$

   Because only the n-th reduced homology group of S^{n-1} is non-trivial, $H_k(M_1 \sharp M_2) \cong H_k(M_1 \vee M_2)$ for $k < n-1$ and we also know that $H_k(M_1 \vee M_2) \cong H_k(M_1) \oplus H_k(M_2)$ for $k \neq 0$. So the only interesting part of the long exact sequence is

$$0 \to H_n(M_1 \sharp M_2) \to H_n(M_1 \vee M_2) \to H_{n-1}(S^{n-1}) \to H_{n-1}(M_1 \sharp M_2) \to H_{n-1}(M_1 \vee M_2) \to 0.$$

   If either M_1 or M_2 is non-orientable, by Part a the map $H_n(M_1 \vee M_2) \to H_{n-1}(S^{n-1})$ is an isomorphism. If both are orientable the map $H_n(M_1 \vee M_2) \to H_{n-1}(S^{n-1})$ sends each fundamental class to a fundamental class of S^{n-1}. Therefore in these cases also $H_{n-1}(M_1 \sharp M_2) \cong H_{n-1}(M_1) \oplus H_{n-1}(M_2)$.
   In the case that both M_1 and M_2 are non-orientable the sequence

$$0 \to H_{n-1}(S^{n-1}) \to H_{n-1}(M_1 \sharp M_2) \to H_{n-1}(M_1 \vee M_2) \to 0 \tag{1}$$

   is exact. By the universal coefficient theorem in homology the sequences

$$0 \to H_n(M_i) \otimes \mathbb{Z}_p \to H_n(M_i; \mathbb{Z}_p) \to \mathrm{Tor}(H_{n-1}(M_i), \mathbb{Z}_p) \to 0$$

   are exact for primes p and $i \in \{1,2\}$. Since $\mathbb{Z}_p$-orientability is the same as $\mathbb{Z}$-orientability for $p \neq 2$, but on the other hand every manifold is $\mathbb{Z}_2$ orientable, the torsion groups of $H_{n-1}(M_i)$ and $H_{n-1}(M_1 \sharp M_2)$ are exactly $\mathbb{Z}_2$. So we can write them as

$$H_{n-1}(M_1) \cong \mathbb{Z}_2 \oplus \mathbb{Z}^a \qquad H_{n-1}(M_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}^b.$$

   By (1) we know that $H_{n-1}(M_1 \sharp M_2)$ has rank $a+b+1$. Therefore it is of the form $\mathbb{Z}^{a+b+1} \oplus T$ for a torsion group T. Since $M_1 \sharp M_2$ is non-orientable we know that $T \cong \mathbb{Z}_2$. Therefore $H_{n-1}(M_1 \sharp M_2)$ is isomorphic to

$$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus (H_{n-1}(M_1)/\operatorname{Tor}(H_{n-1}(M_1))) \oplus (H_{n-1}(M_2)/\operatorname{Tor}(H_{n-1}(M_2)))$$

(c) Compute the cup product structure in $H^*((S^2 \times S^8)\sharp(S^4 \times S^6);\mathbb{Z})$, and in particular show that the only nontrivial cup products are those dictated by Poincaré duality.

Solution: We know that additively, $H^k((S^2\times S^8)\sharp(S^4\times S^6);\mathbb{Z})$ is isomorphic to $\mathbb{Z}$ if $k \in \{0, 2, 4, 6, 8, 10\}$ and zero otherwise. Let e_k for $k \in \{0, 2, 4, 6, 8, 10\}$ be a generator of H^k. The two projections induce a map

$$H^k((S^2 \times S^8) \sqcup (S^4 \times S^6);\mathbb{Z}) \to H^k((S^2 \times S^8)\sharp(S^4 \times S^6);\mathbb{Z})$$

which is an isomorphism in degrees k different from 0 and 10. Therefore the only non-trivial cup products not involving e_0 are up to symmetry $e_2 \cup e_8$ and $e_4 \cup e_6$. Because of Poincaré duality each of these have to be mapped to a fundamental class. (Which one depends on the choice of generators e_i.