Exercise sheet 1

1. Let K be a field. For each of the following statements, indicate whether it is true (with a proof) or false (by giving and explaining a counterexample):

1. Every algebraic extension L of K is a finite extension.
2. The field $\mathbb{C}$ is an algebraic closure of $\mathbb{Q}$.
3. Let L/K be a finite extension and $x \in L$; if P is the minimal polynomial of x, then we have $[L : K] = \deg(P)$.
4. The separable degree of the extension $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$ is 4.
5. There exists a finite field of order 243.
6. The extension $\mathbb{Q}(\exp(2i\pi/123))/\mathbb{Q}$ is algebraic.
7. If K_2/K_1 and K_1/K are algebraic extensions, then K_2/K is algebraic.
8. Let $L = \mathbb{Q}(\sqrt{2}, \exp(2i\pi/127), \sqrt{3 + \sqrt[4]{12}})$; there exists $x \in \mathbb{C}$ such that $L = \mathbb{Q}(x)$.
9. Let L/K be a separable field extension and $n \geq 1$ an integer such that $[K(x) : K] \leq n$ for all $x \in L$; then $[L : K] \leq n$.

2. Let $x = \sqrt{2} + \sqrt[3]{3}$.

1. Prove that $\mathbb{Q}(x) = \mathbb{Q}(\sqrt{2}, \sqrt[3]{3})$. [*Hint:* Find the minimal polynomial of $x - \sqrt{2}$ and expand]
2. Compute the minimal polynomial of x over $\mathbb{Q}(\sqrt{2})$. [*Hint:* $[\mathbb{Q}(x) : \mathbb{Q}(\sqrt{2})] =?$]
3. Compute the minimal polynomial of x over $\mathbb{Q}$.

3. Let p be a prime number and K a field of characteristic p. Let $\phi : K \to K$ be the Frobenius morphism given by $\phi(x) = x^p$.

1. Give an example of field K where ϕ is surjective, and an example where it is not.

We assume that ϕ is surjective.

2. Let $P \in K[X]$ be a polynomial such that $P' = 0$. Prove that there exists $Q \in K[X]$ such that $P = Q^p$.
3. Deduce that any irreducible polynomial $P \in K[X]$ is separable.
4. Deduce that any algebraic extension L/K is separable.

Please turn over!

4. Find an element $x \in K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ such that $K = \mathbb{Q}(x)$.

5. Let K be a field and let E_1 and E_2 be two algebraically closed extensions of K. Let $\bar{K}_1$ and $\bar{K}_2$ denote the algebraic closure of K in E_1 and E_2 respectively.

Let L be an algebraic extension of K.

1. Show that for any field homomorphism $\sigma : L \to E_1$ such that $\sigma|_K = \mathrm{Id}_K$, the image $\sigma(L)$ is contained in $\bar{K}_1$.
2. Show that the number of field homomorphisms $\sigma : L \to E_1$ such that $\sigma|_K = \mathrm{Id}_K$ is equal to the number of field homomorphisms $\sigma : L \to E_2$ such that $\sigma|_K = \mathrm{Id}_K$.