Exercise sheet 11

1. Let

$$\varrho : G \to \mathrm{GL}(V)$$

be a K-representation of a group G. Let $E = \mathrm{End}(V)$ be the vector space of linear maps from V to V.

1. Show that defining
$$\tau(g)A = gAg^{-1}$$
defines a representation τ of G on E.
2. Show that E^G, the space of fixed points of E for this representation, is equal to $\mathrm{Hom}_G(V, V)$.

2. Let

$$\varrho : G \to \mathrm{GL}(V)$$

be a K-representation of a group G, and let

$$\chi : G \to K^\times$$

be a one-dimensional representation.

1. Show that defining
$$\varrho_\chi(g) = \chi(g)\varrho(g)$$
gives a representation ϱ_χ of G on V.
2. Show that a subspace W of V is stable under ϱ if and only if it is stable under ϱ_χ.
3. Show that ϱ is irreducible (resp. semisimple) if and only if ϱ_χ is irreducible (resp. semisimple).

3. Let $G = \mathbb{C}$, $V = \mathbb{C}^2$ and define ϱ by

$$\varrho(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(V).$$

1. Show that ϱ is a representation of G on V.
2. Show that the line $L \subset V$ spanned by the first basis vector is a subrepresentation of G.

Please turn over!

3. Show that there *does not* exist a subspace $W \subset V$ such that $L \oplus W = V$ and W is a subrepresentation.
4. Show that ϱ is *not* semisimple.

4. Let

$$\varrho : G \to \mathrm{GL}(V)$$

be a K-representation of a group G. Let V' be the dual vector space to V.

1. Define $\pi(g) \in \mathrm{End}(V')$ by the relation

$$(\pi(g)(\lambda))(v) = \lambda(\varrho(g^{-1})(v))$$

for $\lambda \in V'$ and $v \in V$. Show that this is a representation of G on V' (it is called the *contragredient* of ϱ).
2. If $\dim(V)$ is finite, find a natural bijection between subrepresentations of ϱ and subrepresentations of π.
3. Deduce that if $\dim(V)$ is finite, then ϱ is irreducible if and only if π is irreducible.
4. If $\dim(V)$ is finite, show that the bidual V'', with the contragredient of the contragredient representation, is isomorphic to V as a representation of G.