Exercise sheet 11

1. Let
$$\varrho : G \to \mathrm{GL}(V)$$
be a K-representation of a group G. Let $E = \mathrm{End}(V)$ be the vector space of linear maps from V to V.

1. Show that defining
$$\tau(g)A = gAg^{-1}$$
defines a representation τ of G on E.
2. Show that E^G, the space of fixed points of E for this representation, is equal to $\mathrm{Hom}_G(V, V)$.

Solution:

1. We need to check that the given formula defines a group homomorphism $\tau : G \to \mathrm{GL}(E)$. This accounts to checking that $\tau(g) : A \mapsto gAg^{-1}$ is a K-linear automorphism of E for each $g \in G$, and that $\tau(gh) = \tau(g)\tau(h)$. Notice that for $v \in V$ and $A \in E$ the formula means
$$(\tau(g)A)(v) := (\varrho(g) \circ A \circ \varrho(g^{-1}))(v).$$
Hence $\tau(g)A \in E$ for each $A \in E$ and $g \in G$, so that $\tau(g) \in \mathrm{End}(E)$. Moreover, multiplicativity of ϱ implies immediately the multiplicative of τ, which at the same time implies that indeed $\tau(g) \in \mathrm{GL}(E)$ for each g and that the resulting $\tau : G \to \mathrm{GL}(E)$ is a group homomorphism.
2. This is an immediate computation:
$$\begin{aligned} E^G &\overset{\text{def}}{=} \{A \in E : \forall g \in G, \tau(g)A = A\} \\ &= \{A \in E : \forall g \in G, \varrho(g) \circ A \circ \varrho(g)^{-1} = A\} \\ &= \{A \in E : \forall g \in G, \varrho(g) \circ A = A \circ \varrho(g)\} \overset{\text{def}}{=} \mathrm{Hom}_G(V, V). \end{aligned}$$

2. Let
$$\varrho : G \to \mathrm{GL}(V)$$
be a K-representation of a group G, and let
$$\chi : G \to K^\times$$
be a one-dimensional representation.

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1. Show that defining
$$\varrho_\chi(g) = \chi(g)\varrho(g)$$
gives a representation ϱ_χ of G on V.
2. Show that a subspace W of V is stable under ϱ if and only if it is stable under ϱ_χ.
3. Show that ϱ is irreducible (resp. semisimple) if and only if ϱ_χ is irreducible (resp. semisimple).

Solution:

1. It is clear that $\chi(g)\varrho(g) \in \mathrm{End}(V)$ for each $g \in G$. It is the endomorphism of V sending $v \mapsto (\chi(g)\varrho(g)) \cdot (v) := \chi(g) \cdot (\varrho(g)(v))$. Moreover, for $g, h \in G$ we see that
$$\varrho_\chi(gh) = \chi(gh)\varrho(gh) = \chi(g)\chi(h)\varrho(g)\varrho(h) = \chi(g)\varrho(g)\chi(h)\varrho(h) = \varrho_\chi(g)\varrho_\chi(h),$$
since constant multiplication commutes with endomorphisms (by definition of linearity). Else ϱ_χ is a representation of G on V.
2. For each $g \in G$ and linear subspace $W \subseteq V$, we have
$$\varrho_\chi(g)(W) = \chi(g)\varrho(g)(W) = \varrho(g)(W),$$
so that W is stable under $\varrho(g)$ if and only it is stable under $\varrho_\chi(g)$. Hence W is stable under ϱ if and only if it is stable under ϱ_χ.
3. The statement concerning irreducibility is immediate from the previous point and the definition of irreducible representation. Moreover, decomposition in direct sums of the two representation correspond (since a decomposition of one of the two representation is just a decomposition of vector spaces $V = W \oplus W'$ where W, W' are stable under the representation, and stability under ϱ and ϱ_χ are equivalent by the previous point).

3. Let $G = \mathbb{C}$, $V = \mathbb{C}^2$ and define ϱ by
$$\varrho(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(V).$$

1. Show that ϱ is a representation of G on V.
2. Show that the line $L \subset V$ spanned by the first basis vector is a subrepresentation of G.
3. Show that there *does not* exist a subspace $W \subset V$ such that $L \oplus W = V$ and W is a subrepresentation.
4. Show that ϱ is *not* semisimple.

Solution:

See next page!

1. Each of the given matrices $\varrho(x)$ is invertible (as it has positive determinant), so that $\varrho(z) \in \mathrm{GL}(V)$. Matrix operations give moreover $\varrho(z)\varrho(w) = \varrho(z+w)$, so that ϱ is indeed a representation of $G = \mathbb{C}$ on V.
2. We take $e_1 = (1,0)$, $e_2 = (0,1)$, so that $V = \mathbb{C} \cdot e_1 + \mathbb{C} \cdot e_2$. Then $L = \langle e_1 \rangle_{\mathbb{C}}$, and for each $z \in G = \mathbb{C}$ we have $\varrho(z)L = \langle \varrho(z)e_1 \rangle = \left\langle \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle = L$, and L is a subrepresentation of G.
3. A subspace W such that $L \oplus W = V$ as complex vector spaces is spanned by any vector $\alpha_1 \cdot e_1 + \alpha_2 \cdot e_2$ with $\alpha_i \in \mathbb{C}$ and $\alpha_2 \neq 0$. It is enough to restrict attention to $\alpha_2 = 1$. For $W_\alpha = \langle v_\alpha := \alpha \cdot e_1 + e_2 \rangle$ to be a subrepresentation of G we need that $\varrho(z)v_\alpha \in W_\alpha$. In particular, we need $\varrho(1)v_\alpha \in W_\alpha$, but
$$\varrho(1)v_\alpha = (\alpha + 1) \cdot e_1 + e_2 = v_\alpha + e_1,$$
which clearly does not lie in W_α. Hence L is a subrepresentation of G, but it is not a direct summand of V as a subrepresentation of G
4. Taking L from the previous point gives a subrepresentation which is not a direct summand of V as a representation of G, showing that V is not semisimple.

4. Let
$$\varrho : G \to \mathrm{GL}(V)$$
be a K-representation of a group G. Let V' be the dual vector space to V.

1. Define $\pi(g) \in \mathrm{End}(V')$ by the relation
$$(\pi(g)(\lambda))(v) = \lambda(\varrho(g^{-1})(v))$$
for $\lambda \in V'$ and $v \in V$. Show that this is a representation of G on V' (it is called the *contragredient* of ϱ).
2. If $\dim(V)$ is finite, find a natural bijection between subrepresentations of ϱ and subrepresentations of π.
3. Deduce that if $\dim(V)$ is finite, then ϱ is irreducible if and only if π is irreducible.
4. If $\dim(V)$ is finite, show that the bidual V'', with the contragredient of the contragredient representation, is isomorphic to V as a representation of G.

Solution:

1. The definition tells us that for each $g \in G$ and $\lambda \in V'$, one has $\pi(g)(\lambda) = \lambda \circ \varrho(g^{-1})$, and this is clearly a K-linear map $V \longrightarrow K$, i.e., an element of V'. Moreover, for $g, h \in G$ we have
$$\begin{aligned}\pi(gh)(\lambda) &= \lambda \circ \varrho(h^{-1}g^{-1}) = \lambda \circ \varrho(h^{-1}) \circ \varrho(g^{-1}) \\ &= \pi(g)(\pi(h))(\lambda),\end{aligned}$$
and this proves that π is indeed a group representation of G on V'.

Please turn over!

2. Let us first find a bijection between subspaces of V and subspaces of V', and then prove that it is compatible with the given representations. Recall that we have a canonical K-linear map

$$\begin{aligned} \gamma : V &\longrightarrow V'' && (1) \\ v &\mapsto \mathrm{ev}_v : (\alpha \mapsto \alpha(v)), && (2) \end{aligned}$$

which is easily seen to be injective. When $\dim(V)$ is finite, this is then an isomorphism of K-vector spaces (as $\dim(V) = \dim(V') = \dim(V'')$). Let us denote by $\mathrm{Sub}(W)$ the set of linear subspaces of W for any K-vector space W. Then we have a map

$$\begin{aligned} \vartheta_V : \mathrm{Sub}(V) &\longrightarrow \mathrm{Sub}(V') && (3) \\ U &\mapsto \mathrm{Ann}_{V'}(U) := \{\alpha \in V' : \alpha(U) = 0\} && (4) \end{aligned}$$

and a bijection induced by γ

$$\begin{aligned} \gamma_* : \mathrm{Sub}(V) &\xrightarrow{\sim} \mathrm{Sub}(V'') && (5) \\ U &\mapsto \gamma(U). && (6) \end{aligned}$$

We claim that $\gamma^{-1} \circ \vartheta_{V'}$ is an inverse of ϑ_V:

a) $\gamma_*^{-1} \circ \vartheta_{V'} \circ \vartheta_V = \mathrm{id}_{\mathrm{Sub}(V)}$: we have to check that for each subspace $U \subseteq V$ one has $\gamma_*^{-1}\mathrm{Ann}_{V''}(\mathrm{Ann}_{V'}(U)) = U$, i.e., $\mathrm{Ann}_{V''}(\mathrm{Ann}_{V'}(U)) = \gamma(U)$, and this is done directly:

$$\begin{aligned} \mathrm{Ann}_{V''}(\mathrm{Ann}_{V'}(U)) &= \{a \in V'' : a(\alpha) = 0, \forall \alpha \in V' : \alpha(U) = 0\} \\ &= \{\gamma(u) : u \in V, \alpha(u) = 0, \forall \alpha \in V' : \alpha(U) = 0\} \\ &= \gamma(U). \end{aligned}$$

Notice that in the last equality the inclusion $\supseteq$ is trivial, while for the other inclusion one can see that for $u' \in V \setminus U$ there is a basis of V obtained by the union of a basis of U with a set of vectors of V which contains u', so that u' can be sent to a non-zero vector by some α which annihilates U.

b) $\vartheta_V \circ \gamma^{-1} \circ \vartheta_{V'} = \mathrm{id}_{\mathrm{Sub}(V')}$: we have to check that for each subspace $U' \subseteq V'$ we have $\mathrm{Ann}_{V'}(\gamma^{-1}(\mathrm{Ann}_{V''}(U'))) = U'$. We indeed have

$$\begin{aligned} \mathrm{Ann}_{V'}(\gamma^{-1}(\mathrm{Ann}_{V''}(U'))) &= \{\alpha \in V' : \alpha(u) = 0, \forall u \in V : \mathrm{ev}_u(U') = 0\} \\ &= \{\alpha \in V' : \alpha(u) = 0, \forall u \in V : u'(u) = 0, \forall u' \in U'\} \\ &= U', \end{aligned}$$

where again the non-trivial inclusion $\subseteq$ is proved similarly as in the previous point.

Then ϑ_V is a bijection $\mathrm{Sub}(V) \xrightarrow{\sim} \mathrm{Sub}(V')$. Notice that for $U' \in \mathrm{Sub}(V')$ we have

$$\begin{aligned} \gamma_*^{-1}\vartheta_{V'}(U') &= \{u \in V : \gamma(u)(U') = 0\} \\ &= \{u \in V : \alpha(u) = 0, \forall \alpha \in U'\} =: \mathrm{Ker}_V(U'). \end{aligned}$$

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It is easily checked that both $\mathrm{Ann}_{V'}$ and $\mathrm{Ker}_{V'}$ reverse inclusions, so that in particular for all $U_1, U_2 \in \mathrm{Sub}(V)$ one has

$$(*) \quad U_1 \subseteq U_2 \iff \mathrm{Ann}_{V'}(U_1) \supseteq \mathrm{Ann}_{V'}(U_2).$$

Let us now check that ϑ_V is compatible with the representations. We have to prove that for $W \in \mathrm{Sub}(V)$ one has that W is fixed by each $\varrho(g)$ if and only if $\mathrm{Ann}_{V'}(W)$ is fixed by each $\pi(g)$.

We have

$$\begin{aligned}
\pi(g)(\mathrm{Ann}_{V'}(W)) \subseteq \mathrm{Ann}_{V'}(W) &\iff \pi(g)(\alpha) \in \mathrm{Ann}_{V'}(W), \forall \alpha \in \mathrm{Ann}_{V'}(W) \\
&\iff \alpha(\varrho(g^{-1})(W)) = 0, \forall \alpha \in \mathrm{Ann}_{V'}(W) \\
&\iff \mathrm{Ann}_{V'}(W) \subseteq \mathrm{Ann}_{V'}(\varrho(g^{-1})W) \\
&\overset{(*)}{\iff} \varrho(g^{-1})W \subseteq W.
\end{aligned}$$

This proves our claim, as $g \mapsto g^{-1}$ is a bijection of G.

3. This is an immediate consequence from the previous point, as the bijection we found sends $V \mapsto 0$ and $0 \mapsto V'$. Recall that irreducibility means that the only subrepresentations are 0 and the whole representation.
4. This just accounts to prove that subrepresentations of G on V and V'' correspond bijectively via γ_*. In point 2, we proved that $\gamma_* = \vartheta_{V'} \circ \vartheta_V$, and that ϑ_V (and hence $\vartheta_{V'}$) restricts to a bijective correspondence of subrepresentations (by taking the contragradient representation on V'). Clearly this property is preserved by composition, whence our claim.