Ferienserie

1. Let G be a finite group.

1. Show that if ϱ is an irreducible representation of G over $\mathbb{C}$, it is finite-dimensional.
2. Show that G is abelian if and only if all irreducible representations of G over $\mathbb{C}$ are one-dimensional. [Hint: to show one implication, apply Schur's Lemma; for the other, count conjugacy classes.]
3. Let ϱ be an irreducible finite-dimensional complex representation of G. Let H be the center of G. Show that there exists a homomorphism $\omega : H \to \mathbb{C}^\times$ such that

$$\varrho(h) = \omega(h)\mathrm{Id}$$

for all $h \in H$ (namely, the center of G acts by scalars).

2. Let G be a finite group, and let

$$\varrho : G \to \mathrm{GL}(V)$$

be a finite-dimensional complex representation of G. Let $E = \mathrm{End}(V)$ be the vector space of linear maps from V to V.

Show that the character of the representation $\tau : G \to \mathrm{GL}(E)$ defined by

$$\tau(g)A = gAg^{-1}$$

is $\chi_\tau(g) = |\chi_\varrho(g)|^2$, where χ_ϱ is the character of ϱ.

3. Let $\varrho : G \to \mathrm{GL}(V)$ be a finite-dimensional complex representation of a finite group G. Consider the linear transformation $P : V \to V$ defined by

$$P(v) = \frac{1}{|G|} \sum_{g \in G} \varrho(g)v.$$

1. Show that P is a linear projection on the subspace V^G of vectors invariant under ϱ (i.e., $P \circ P = P$, and the image of P is V^G).
2. Show that $P \in \mathrm{Hom}_G(V, V)$.
3. Let $\langle \cdot, \cdot \rangle$ be an inner-product on V such that $\varrho(g)$ is unitary with respect to the inner-product for all g. Show that P is the orthogonal projection on V^G.

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4. Show that the dimension of V^G is given by

$$\dim V^G = \sum_{g \in G} \chi_\varrho(g)$$

where χ_ϱ is the character of G.

4. Let G be a finite group. Let $n \geq 1$, and suppose that G acts on the finite set $X_n = \{1, \ldots, n\}$. Let $V = \mathbb{C}^n$ and $(e_1, \ldots, e_n)$ the canonical basis of V.

1. Show that if we define $\varrho(g) : V \to V$ by

$$\varrho(g) e_i = e_{g \cdot i},$$

we obtain a representation of G on V.

2. Show that the character of ϱ is given by

$$\chi_\varrho(g) = |\{i \in X_n \mid g \cdot i = i\}|.$$

3. Assume that G acts transitively on X_n. Show that

$$\dim V^G = 1,$$

and identify a generator of V^G.

4. Show that the subspace

$$W = \{(x_1, \ldots, x_n) \in V \mid x_1 + \cdots + x_n = 0\}$$

is a subrepresentation of V. Let π be the representation of G on W. Show that

$$\chi_\pi(g) = \chi_\varrho(g) - 1.$$

Assume now that $n \geq 2$ and that G acts doubly-transitively on X_n: for any $i \neq j$ in X_n, there exists $g \in G$ such that $g \cdot 1 = i$ and $g \cdot 2 = j$.

5. Show that for any $i \neq j$, the subset

$$\{g \in G \mid g \cdot 1 = i \text{ and } g \cdot 2 = j\}$$

has the same size.

6. Show that

$$\frac{1}{|G|} \sum_{g \in G} |\chi_\pi(g)|^2 = 1,$$

and deduce that the action of G on W is irreducible.