Exercise sheet 2

1. Let k be a field with $\mathrm{char}(k) \neq 2$.

1. Let $a, b \in k$ be such that a is a square in $k(\beta)$, where β is an element algebraic over k such that $\beta^2 = b$. Prove that either a or ab is a square in k. [*Hint:* Distinguish the cases $\beta \in k$ and $\beta \notin k$. For the second case, expand $(c + d\beta)^2$, for $c, d \in k$.]
2. Now consider $K = k(u, v)$, where $u, v \notin k$ are elements in an algebraic extension of k such that $u^2, v^2 \in k$. Set $\gamma = u(v + 1)$. Prove: $K = k(\gamma)$.

2.

1. Prove that if $[K : k] = 2$, then $k \subseteq K$ is a normal extension.
2. Show that $\mathbb{Q}(\sqrt[4]{2}, i)/\mathbb{Q}$ is normal.
3. Show that $\mathbb{Q}(\sqrt[4]{2}(1 + i))/\mathbb{Q}$ is not normal over $\mathbb{Q}$.
4. Deduce that given a tower $L/K/k$ of field extensions, L/k needs not to be normal even if L/K and K/k are normal.

3. Let K be a field, and $L = K(X)$ a field of rational functions.

1. Show that, for any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(K)$, the map
$$\sigma_A(f) = f\left(\frac{aX + b}{cX + d}\right)$$
defines a K-automorphism of L, and we obtain a group homomorphism
$$i : \mathrm{GL}_2(K) \longrightarrow \mathrm{Aut}(L/K).$$
2. Compute $\ker(i)$.
3. For $f \in K(X)$, write $f = \frac{p(X)}{q(X)}$, with $p(X), q(X) \in K[X]$ coprime polynomials. Prove that $p(X) - q(X)Y$ is an irreducible polynomial in $K[X, Y]$, and deduce that X is algebraic of degree $\max\{\deg(p), \deg(q)\}$ over $K(f)$.
4. Conclude that i is surjective [*Hint:* For $\sigma \in \mathrm{Aut}(L/K)$, apply previous point with $f = \sigma(X)$].
5. Is an endomorphism of the field $K(X)$ which fixes K always an automorphism?

Please turn over!

4.

1. Let K be field containing $\mathbb{Q}$. Show that any automorphism of K is a $\mathbb{Q}$-automorphism.
2. From now on, let $\sigma : \mathbb{R} \longrightarrow \mathbb{R}$ be a field automorphism. Show that σ is increasing:

$$x \leq y \Longrightarrow \sigma(x) \leq \sigma(y).$$

3. Deduce that σ is continuous.
4. Deduce that $\sigma = \mathrm{Id}_{\mathbb{R}}$.