Exercise sheet 3

1. Let L/K be a Galois extension, and $G = \mathrm{Gal}(L/K)$. Fix $x \in L$ and let $f(X) = \mathrm{Irr}(x, K)(X)$. Show that we have the following equality of subsets in L:

$$\{\sigma(x) | \sigma \in G\} = \{\alpha \in L : f(\alpha) = 0\}.$$

2. Let L be a field, $G \subseteq \mathrm{Aut}(L)$ a finite subgroup of cardinality n, and consider the subfield $K = L^G$ of L. Prove:

1. L/K is a finite extension of degree n [*Hint:* Exercise 1.9 from Exercise sheet 1].
2. L/K is Galois with group G.

3. Let L/K be a finite extension. Prove that L/K is Galois if and only if $|\mathrm{Aut}_K(L)| = [L : K]$. [You can apply the primitive element theorem]

4. Let K be an infinite field, and V a K-vector space over K. Prove that if $V_1, \ldots, V_m$ are vector subspaces in V such that $V_i \neq V$ for all i, then $\bigcup_{i=1}^m \neq V$. [*Hint:* Induction on n].

5. In this exercise, we will show how to prove the primitive element theorem using Galois theory. This is useful because it is possible to prove the Galois correspondence without the primitive element theorem, see Section 4.3 in Reid's notes.

Let L/K be a finite separable field extension.

1. Prove that there exist only finitely many intermediate field extensions $K \subseteq E \subseteq L$. [You can use the fact that L embeds in a Galois closure L^g, that is, a smallest finite extension of L such that $K \subseteq L^g$ is Galois]
2. Deduce that if K is an infinite field, then $L = K(x)$ for some $x \in L$. [*Hint:* Previous exercise]
3. Suppose that K is finite. Prove that $L = K(x)$ for some $x \in L$.