Exercise sheet 4

1. Let K be a field of characteristic 2, and fix an algebraic closure $\bar{K}$ of K. Suppose L/K is a Galois quadratic extension contained in $\bar{K}$.

1. Show that there exists $a \in K$ such that $L = K(b)$ where b is a root of $X^2 - X + a$.
2. Prove that $\mathrm{Gal}(L/K) \cong \mathbb{Z}/2\mathbb{Z}$, and express the action of the generator of G on L as a matrix with respect to the basis $(1, b)$.
3. Suppose that for $i = 1, 2$ we have elements $a_i \in K$ and we consider the field extensions $L_i = K(b_i)$, where $b_i \in \bar{K}$ are roots of polynomials $X^2 - X + a_i$, which we suppose to be irreducible. Show that $L_1 = L_2$ if and only if there exists $\mu \in K$ such that $\mu^2 - \mu = a_2 - a_1$.

2. Consider the polynomial $f = X^3 - 2 \in \mathbb{Q}[X]$, and let L be the splitting field of f.

1. Prove that $[L : \mathbb{Q}] = 6$, and find intermediate extensions L_1 and L_2 of L over $\mathbb{Q}$ such that $[L_1 : \mathbb{Q}] = 2$ and $[L_2 : \mathbb{Q}] = 3$.
2. Prove that $L/\mathbb{Q}$ is a Galois extension with Galois group $G = S_3$ [*Hint:* The Galois group of L acts faithfully on the roots of f].
3. Which of the four field extensions L/L_i and $L_i/\mathbb{Q}$, for $i = 1, 2$ are Galois? Find their Galois groups.

3. Let K be a field and $P \in K[X]$ a separable degree-n irreducible polynomial, L its splitting field and $G = \mathrm{Gal}(L/K)$.

0. Prove that $|G| \leq \deg(P)!$

From now on, assume that P is a palindromic monic polynomial of even degree, i.e., there exist a positive integer d and elements $a_1, \ldots, a_d$ such that

$$P = X^{2d} + a_1 X^{2d-1} + \cdots + a_{d-1} X^{d+1} + a_d X^d + a_{d-1} X^{d-1} + \cdots + a_1 X + 1.$$

Show that:

1. The set of roots Z_P of P is stable under $x \mapsto \frac{1}{x}$.
2. Given the following subgroup of $S_{2d} = \mathrm{Sym}(\{\alpha_1^+, \alpha_1^-, \alpha_2^+, \alpha_2^-, \ldots, \alpha_d^+, \alpha_d^-\})$:
   $$W_{2,d} = \{\sigma \in S_{2d} | \, \forall i \, \exists j : \sigma(\{\alpha_i^+, \alpha_i^-\}) = \{\alpha_j^+, \alpha_j^-\}\},$$
   we have that G can be embedded in $W_{2,d}$.
3. $|G| \leq 2^d d!$

Please turn over!

4. Let $K = \mathbb{Q}[\sqrt{2}, \sqrt{3}]$.

1. Show that K is Galois over $\mathbb{Q}$ with Galois group the $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
2. Now let $L = K\left[\sqrt{(\sqrt{2}+2)(\sqrt{3}+3)}\right]$. Show that L is Galois over $\mathbb{Q}$.

5. Let L/K be a finite Galois extension. Take $x \in L$ and assume that the elements $\sigma(x)$ are all distinct for $\sigma \in \mathrm{Gal}(L/K)$. Show: $L = K(x)$.