Exercise sheet 5

Reminders. Let G be a group. We call a G-*set* a set X endowed with an action of G. We say that a map $f : X \longrightarrow Y$ between two G-sets is G-*equivariant* (or a G-*map*, or a map of G-*sets*) if for each $g \in G$ and $x \in X$ one has $g \cdot f(x) = f(g \cdot x)$, where the dot is used for the actions of G on both X and Y. An *isomorphism of* G-*sets* is a G-map $X \longrightarrow Y$ which has a two-sided inverse G-map $Y \longrightarrow X$. It is easy to see that compositions of G-maps are G-maps, and that $f : X \longrightarrow Y$ is an isomorphism of G-sets if and only if it is a bijective G-map.

The content of the marked exercise **(*)** should be known for the exam.

1. Let L/K be a finite Galois extension with Galois group G. Fix an algebraic closure $\bar{K}$ of K containing L and consider an intermediate extension $L/E/K$.

1. Prove that composition of field homomorphisms induces an action of G on the set of K-embeddings $E \longrightarrow \bar{K}$.
2. Let τ_0 be the inclusion $E \hookrightarrow \bar{K}$, and take $H = \mathrm{Stab}_G(\tau_0)$. Prove that $H = \mathrm{Gal}(L/E)$ and deduce that $L^H = E$.
3. Now assume that L is the splitting field of an irreducible separable polynomial $P \in K[X]$, and that $E = K(x_0)$ for some root x_0 of P. Show that the set of K-embeddings $E \longrightarrow \bar{K}$ is isomorphic as a G-set to the set Z_P of roots of P with the usual action of G.

2. **(*)** Let L/K be a finite Galois extension of degree n with Galois group G. For $x \in L$, let m_x be the K-linear map $L \longrightarrow L$ sending $y \mapsto xy$. We define the trace and the norm maps $\mathrm{Tr}_{L/K}, \mathrm{N}_{L/K} : L \longrightarrow K$ as

$$\mathrm{Tr}_{L/K}(x) = \mathrm{Tr}(m_x) \quad \text{and} \quad \mathrm{N}_{L/K}(x) = \det(m_x).$$

[See Exercise sheet 11 from Algebra I, HS14]

1. Let $x \in L$. Denote $\chi_x(X) \in K[X]$ the characteristic polynomial of m_x, and $d_x = [K(x) : K]$. Prove: $\chi_x = (\mathrm{Irr}_{x;K})^{n/d}$.
2. Show that for each $x \in L$ we have

$$\mathrm{Tr}_{L/K}(x) = \sum_{\sigma \in G} \sigma(x) \quad \text{and} \quad \mathrm{N}_{L/K}(x) = \prod_{\sigma \in G} \sigma(x).$$

Please turn over!

3. Show that if $M/L/K$ is a tower of Galois extensions, then $\mathrm{N}_{M/K} = \mathrm{N}_{L/K} \circ \mathrm{N}_{M/L}$.

Notice that the last property in fact holds for any tower of finite extension, but the proof is more complicated.

3. Let L/K be a finite Galois extension with Galois group G.

1. Prove that the action of G on $L[X]$ (as seen in class) extends to an action on the field of rational functions $L(X)$ via $\sigma \cdot \left(\frac{P}{Q}\right) = \frac{\sigma(P)}{\sigma(Q)}$.
2. Check that $L(X)^G = K(X)$.

4. For any field K, we consider the projective line

$$\mathbb{P}(K) := (K^2 \setminus \{0\})/\sim,$$

where $(a, b) \sim (c, d)$ if there exists $\lambda \in K^\times$ such that $(c, d) = (a\lambda, b\lambda)$.

1. Check that $\sim$ is indeed an equivalence relation.
2. Prove that for any field extension L/K the map $(x, y) \mapsto (x, y)$ induces an injection $j : \mathbb{P}(K) \hookrightarrow \mathbb{P}(L)$.

From now on, assume that L/K is a finite Galois extension with Galois group G.

3. Prove that $\sigma \cdot (a, b) = (\sigma(a), \sigma(b))$ gives a well-defined action of G on $\mathbb{P}(L)$.
4. Check that $\mathbb{P}(L)^G$ is the image of $\mathbb{P}(K)$ via the injection j.

5. Let $f \in \mathbb{Q}[X]$ be a monic polynomial of degree $n > 2$, and L_f its splitting field over $\mathbb{Q}$. Let $G_f = \mathrm{Gal}(L/K)$, and suppose that the inclusion $G_f \hookrightarrow S_n$ is an isomorphism.

1. Show that f is irreducible over $\mathbb{Q}$
2. Given a root α of f, prove that the only automorphism of the field $\mathbb{Q}(\alpha)$ is the identity.