Answers to Multiple Choice Questions

1. Let L/K be a finite extension of fields. Which of the following assertions are correct:

A. If the characteristic of K is zero, then L/K is normal.
B. If the characteristic of K is zero, then L/K is separable.
C. If L/K is normal, then L/K is a Galois extension.
D. If the characteristic of K is positive, then L/K is normal if and only if it is separable.

Answer:

- (A) is not correct (counterexample: $\mathbf{Q}(\sqrt[3]{2})/\mathbf{Q}$ is not normal).
- (B) is correct (result from 1st semester)
- (C) is not correct (if L/K is not separable; counterexample $\mathbf{F}_p(T^{1/p})/\mathbf{F}(T)$).
- (D) is not correct (counterexample: $\mathbf{F}_p(T^{1/p})/\mathbf{F}_p(T)$ is normal but not separable).

2. Let L/K be a finite extension of fields. Which of the following assertions are correct:

A. If $L = K(x)$, where x is a root of a separable polynomial in $K[X]$, then L/K is separable.
B. There exists $x \in L$ such that $L = K(x)$.
C. For any embedding σ of K in an algebraic closed-field $\bar{K}$, there exists $\tau : L \to \bar{K}$ which extends σ.

Answer:

- (A) is correct (result from 1st semester)
- (B) is not correct in general (result from 1st semester, example is $\mathbf{F}_p(X^{1/p}, Y^{1/p})$.)
- (C) is correct (result from 1st semester).

3. Is it true that if K is a finite field, then any finite extension L/K is a Galois extension? What about any algebraic extension?

Answer: This is correct because any finite extension of K is a finite field, and any extension of finite fields is Galois by a result from the class. This is not the case for algebraic extensions with the definition in class because such extensions may be of infinite degree. (With proper definitions, in fact, any algebraic extension of a finite field is Galois).

Please turn over!

4. Let K be a field, $\bar{K}$ an algebraic closure of K and $P \in K[X]$ a non-constant polynomial. Let $L \subset \bar{K}$ denote the splitting field of P in $\bar{K}$. Which of the following assertions are correct:

A. The extension L/K is a normal extension.

B. If $x \in \bar{K}$ is a root of P, then $L = K(x)$.

C. The extension L/K is a Galois extension.

D. If the polynomial P is irreducible, then L/K is a Galois extension.

E. If the characteristic of K is zero, then L/K is a Galois extension.

Answer:

- (A) is correct (one of basic example of normal extension)
- (B) is not correct, because a single root of P might not be enough (counterexample: $K = \mathbf{Q}$, $P = X^3 - 2$; then $\mathbf{Q}(\sqrt[3]{2})$ is not the splitting field of P).
- (C) is not always correct (only if P is separable; counterexample is $K = \mathbf{F}_p(T)$, $P = X^p - T$).
- (D) is not always correct (only if P is separable; same counterexample).
- (E) is correct (because L/K is always separable in that case).

5. Let K be a field, $\bar{K}$ an algebraic closure of K and $L \subset \bar{K}$ a finite extension of K such that L/K is a Galois extension. Let $K \subset E \subset L$ be an intermediate extension. Which of the following assertions are correct:

A. The extension L/E is a Galois extension.

B. The extension E/K is a normal extension.

C. The extension E/K is a separable extension.

Answer:

- (A) is correct (basic result from Galois correspondance)
- (B) is not correct (counterexample: $K = \mathbf{Q}$, L splitting field of X^3-2, $E = \mathbf{Q}(\sqrt[3]{2})$; the $E/\mathbf{Q}$ is not normal).
- (C) is correct (subextensions of separable extensions are separable, as follows for instance from the characterization using separability of minimal polynomials).

6. Let K be a field, $\bar{K}$ an algebraic closure of K and $L \subset \bar{K}$ a finite extension of K such that L/K is a Galois extension, and let G be its Galois group. Which of the following assertions are correct:

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A. For any subgroup H of G, the intermediate extension $E = L^H$ is a normal extension of K.

B. Two subgroups H_1 and H_2 of G are equal if and only if $L^{H_1} = L^{H_2}$.

C. Any subgroup H of G is the Galois group of some extension E/K for some $E \subset L$.

D. Any subgroup H of G is the Galois group of some extension L/E for some $E \subset L$.

Answer:

- (A) is not correct ($E = L^H$ is normal over K if and only if H is a normal subgroup of K)
- (B) is correct (injectivity of the map $H \mapsto L^H$ in the Galois correspondance)
- (C) is not correct (counterexample: if $G = S_3$ is the symmetric group and H is generated by a cycle of length 3, so that H has order 3, then an intermediate E with $\mathrm{Gal}(E/K) = H$ would correspond to a normal subgroup $K < G$ with $[S_3 : K] = [L : E] = 2$, but one can see easily that there is no normal subgroup of order 2 in S_3)
- (D) is correct (Galois correspondance: one can take $E = L^H$ since $H = \mathrm{Gal}(L/L^H)$)

7. Let K be a field, $\bar{K}$ an algebraic closure of K and $L \subset \bar{K}$ a finite extension of K such that L/K is a Galois extension, and let G be its Galois group. Let $x \in L$ be given and $\sigma_0 \in G$ a non-trivial element. Which of the following assertions are correct:

A. If $\sigma_0(x) = x$, then $x \in K$.

B. If G is cyclic and $\sigma_0(x) = x$, then $x \in K$.

C. The element

$$\sum_{\sigma \in G} \sigma(x)^2$$

belongs to K.

D. If the set of all $\sigma(x)$, for σ ranging over G, contains at most two elements, then $[K(x) : K] \leq 2$.

Answer:

- (A) is not correct (by Galois correspondance, $x \in K$ if and only if $\sigma(x) = x$ for all $\sigma \in G$; so $\sigma_0(x) = x$ does not imply $x \in K$ unless σ_0 generates G)
- (B) is not correct (although G is cyclic, it might be that σ_0 is not a generator)
- (C) is correct (by Galois correspondance, one checks by reordering the sum that the sum y indicated satisfies $\tau(y) = y$ for all $\tau \in G$, so that $y \in L^G = K$).
- (D) is correct (the assumption implies that the separable degree of $K(x)/K$ is at most 2, since the roots of the minimal polynomial P of x are among the values $\sigma(x)$, by transitivity of the action of the Galois group of the splitting field of P on the set of roots).

Please turn over!

8. Let K be a field, $\bar{K}$ an algebraic closure of K and $L \subset \bar{K}$ a finite extension of K of degree 2. Which of the following assertions are correct:

A. The extension L/K is separable.

B. The extension L/K is normal.

C. If the characteristic of K is zero, then there exists $y \in L$ such that $L = K(y)$ and $y^2 \in K^\times$.

Answer:

- (A) is not correct (counterexample if $\mathbf{F}_2(\sqrt{T})/\mathbf{F}_2(T)$)
- (B) is correct (result from the class)
- (C) is correct (result from the class)