Exercise sheet 8

1. In this exercise, we will give a characterization for solvable groups using commutator subgroups. See last semester's (Algebra I, HS 2014) Exercise Sheet 3, Exercise 6, for the definition and some properties of the commutator subgroup.

1. Let G be a group and $G_1 \trianglelefteq G$ a normal subgroup such that G/G_1 is abelian. Show that
$$[G,G] \subseteq G_1.$$
2. Deduce that G is solvable if and only if there exists $m \geq 1$ such that $G^{(m)} = \{1\}$, where the $G^{(m)}$ are subgroups defined inductively via
$$\begin{aligned} G^{(0)} &= G \\ G^{(i+1)} &= [G^{(i)}, G^{(i)}]. \end{aligned}$$

2.
1. Show that S_3 and S_4 are solvable groups.
2. Show that the group A_5 is generated by the two permutations $(1\ 2)(3\ 4)$ and $(1\ 3\ 5)$.
3. Show that $[S_5, S_5] = A_5$ and deduce that the group S_5 is not solvable.

3. Let K be a field and consider the group
$$B_2 = \left\{ \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \,\middle|\, a, b \in K^\times, x \in K \right\} \leq \mathrm{GL}_2(K).$$
Show that
$$[B_2, B_2] = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \,\middle|\, x \in K \right\}.$$
Can you find a generalization to the subgroup B_n of upper-triangular matrices in $\mathrm{GL}_n(K)$, for $n \geq 2$?

4. *(Gauss's Lemma)* Let R be a UFD and $K = \mathrm{Frac}(R)$. We say that the elements $a_1, \ldots, a_n \in R$ are *coprime* if whenever $u | a_i$ for each i, then $u \in R^\times$. We call a non-zero polynomial $p \in R[X]$ *primitive* if its coefficients are coprime. Prove the following statements:

Please turn over!

1. Each irreducible element in R (i.e., a non-zero non-unit in R which cannot be written as product of two non-units) is prime in R (i.e., whenever it divides a product bc, then it divides b or c).
2. If $a, b \in R$ are coprime and $b|ac$ for some $c \in R$, then $b|c$.
3. Any element $\lambda \in K$ can be written as a quotient $\lambda = a/b$, with $a, b \in R$ coprime elements.
4. The product of two primitive polynomials $p, q \in R[X]$ is a primitive polynomial. [*Hint:* For d an irreducible element, notice that there is an isomorphism of rings $R[X]/dR[X] \cong (R/dR)[X]$, and deduce that $R[X]/dR[X]$ is an integral domain.]
5. If $f \in R[X]$ can be factored as $f = gh$ with $g, h \in K[X]$, then there exist $g', h' \in R[X]$ such that $f = g'h'$ and $g = \lambda g'$ for some $\lambda \in K$. [*Hint:* Prove that one can write $g = \gamma \cdot G$ for some $\gamma \in K$ and $G \in R[X]$ primitive polynomial. You main need to use the three previous points.]
6. A polynomial $f \in R[X]$ is irreducible in $R[X]$ if and only if it is primitive and it is irreducible in $K[X]$.

The last three statements are usually referred to as Gauss's Lemma.