Exercise sheet 9

1. Let G be a solvable group, and H a subgroup of G, not necessarily normal. Prove that H is solvable.

2. The aim of this exercise is to explain Cardan's formula for solutions of a degree-3 polynomial equation.

Let K be a field of characteristic 0 and $P \in K[X]$ be an irreducible degree 3 polynomial. Denote by L the splitting field of P, and assume that $\mathrm{Gal}(L/K) = S_3$. Up to a change of variable, we can assume that $P(X) = X^3 + pX + q$. Then one can find that the discriminant of P is $\Delta = -4p^3 - 27q^2$.

1. Show that Δ is not a square in K, and that $[L : K(\Delta)] = 3$.
2. Let μ_3 be the group of cubic roots of 1 in $\bar{K}$. Show that $L(\mu_3)/K(\sqrt{\Delta}, \mu_3)$ is a Galois extension of degree 3. Deduce that $\mathrm{Gal}(L(\mu_3)/K(\sqrt{\Delta}, \mu_3)) \cong \mathbb{Z}/3\mathbb{Z}$. [*Hint:* $[K(\sqrt{\Delta}, \mu_3) : K(\sqrt{\Delta})] \leq 2$.]
3. Let σ be a generator of $\mathrm{Gal}(L(\mu_3)/K(\sqrt{\Delta}, \mu_3)) \cong \mathbb{Z}/3\mathbb{Z}$, and x a root of P in L. Prove that the set of roots of P in L is $\{x, \sigma(x), \sigma^2(x)\}$.
4. Let $\xi \in \bar{K}$ be a primitive cubic root of unity, and consider the Lagrange resolvents
$$\alpha := x + \xi\sigma(x) + \xi^2\sigma^2(x)$$
$$\beta := x + \xi^2\sigma(x) + \xi\sigma^2(x).$$
Prove that $x, \sigma(x), \sigma^2(x)$ can be expressed in terms of α and β. Deduce that α and β are non-zero and that $L(\mu_3) = K(\sqrt{\Delta}, \mu_3, \alpha)$. [*Hint:* $x + \sigma(x) + \sigma^2(x) = 0$. Use linear systems.]
5. Explain why α^3 and β^3 belong to $K(\sqrt{\Delta}, \mu_3)$. Why does this allow to solve the cubic in principle?
6. From now on denote the three roots of P as x_1, x_2 and x_3. Consider $D = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$, so that $D^2 = \Delta$. Define also
$$A := x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_1$$
$$B := x_1 x_2^2 + x_2 x_3^2 + x_3 x_1^2.$$
Prove the following equalities
$$\alpha^3 = -9q + 3\xi A + 3\xi^2 B, \quad \beta^3 = -9q + 3\xi^2 A + 3\xi B$$
Find A, B in terms of D and use this to find α and β. [*Hint:* See Chambert-Loir, *A field guide to algebra*, page 121, for further hints.]