Test exam

1. (Groups)

1. (Result from the course) Prove that if H is a normal subgroup of a group G, there is a group structure on the set G/H of right H-cosets of G such that the projection map $\pi : G \to G/H$ is a homomorphism. Prove that a homomorphism $\varphi : G \to G_1$, where G_1 is another arbitrary group, can be expressed in the form $\varphi = \tilde{\varphi} \circ \pi$ for some homomorphism $\tilde{\varphi} : G/H \to G_1$ if and only if $\ker(\varphi) \subset H$.
2. Which of the following statements are true (justify with a proof, a reference to a result of the course, or a counterexample):
   A. Every finite abelian group is isomorphic to a direct product of cyclic groups.
   B. Every subgroup of an abelian group is solvable.
   C. If a group G acts on a set X, then the stabilizer of a point $x \in X$ is a normal subgroup of G.
3. Let G be a group, H a subgroup of G and $\xi \in G$ an element such that $\xi H \xi = H$. Prove that $\xi^2 \in H$ and that $\xi H \xi^{-1} = H$ (which means that ξ belongs to the normalizer of H in G). Conversely, prove that if $\eta \in G$ is some element such that $\eta^2 \in H$ and $\eta \in N_G(H)$, then $\eta H \eta = H$.

2. (Rings)

1. (Result from the course) Prove that in a principal ideal domain A, every non-zero element has a unique factorization into irreducible elements.
2. State the structure theorem for finitely-generated modules over a principal ideal domain.
3. Which of the following statements are true (justify with a proof, a reference to a result of the course, or a counterexample):
   A. If I and J are ideals in a commutative ring A, then $A/(I \cap J)$ is isomorphic to $A/I \times A/J$.
   B. Any integral domain A is contained in a field K.
   C. Any non-zero commutative ring contains a prime ideal.
   D. If A is a commutative ring and $I \subset A$ is a prime ideal, then A/I is a field.
4. Let K be a field and $n \geq 2$ an integer. Let I_n denote the principal ideal generated by X^n in $K[X]$, and let $A_n = K[X]/I_n$. Compute the group $A_n^\times$ of units in A_n. Prove that A_n has a unique maximal ideal; which ideal is it?

Please turn over!

3. (Fields)

1. (Result from the course) Prove that given a field K and a non-constant polynomial $P \in K[X]$, there exists an extension L/K and an element $x \in L$ such that $P(x) = 0$.
2. Which of the following statements are true (justify with a proof, a reference to a result of the course, or a counterexample):

   A. If L/K is a finite extension and L contains some element x for which the minimal polynomial $\mathrm{Irr}(x; K)$ of x is separable, then L/K is separable.

   B. If K is a finite field, then its order is a prime number.

   C. If K is a field and L_1, L_2 are algebraically closed fields containing K, then L_1 is isomorphic to L_2.

4. (Galois theory)

1. (Result from the course) Given a field K, a separable non-constant polynomial $P \in K[X]$ of degree $d \geq 1$ and a splitting field L/K of P, explain the construction of an injective homomorphism $\mathrm{Gal}(L/K) \to S_d$.
2. (Result from the course) State and sketch the proof of the classification of Kummer extensions for cyclic extensions of degree d over a field K containing the d-th roots of unity.
3. Which of the following statements are true (justify with a proof, a reference to a result of the course, or a counterexample):

   A. If L/K is a finite extension of finite fields, then L/K is a Galois extension.

   B. For any field K of characteristic 0, any $n \geq 2$, and $L = K(y)$ where $y^n = 2$, the extension L/K is a Galois extension.

   C. Any radical extension has a solvable Galois group.
4. Let L/K be a finite Galois extension with Galois group G. Let G' denote the commutator subgroup $[G, G]$ generated by all commutators $xyx^{-1}y^{-1}$ in G. Show that $L^{G'}/K$ is a Galois extension with $\mathrm{Gal}(L^{G'}/K)$ abelian. Show that any Galois extension E/K with $E \subset L$ and $\mathrm{Gal}(E/K)$ abelian is contained in $L^{G'}$.
5. Let K be a field of characteristic zero, and let $\bar{K}$ be an algebraic closure of K. Let x and y be elements of $\bar{K}$ such that $K(x)$ and $K(y)$ are solvable extensions. Prove that $K(x + y)$ is also solvable.