ETH Zürich FS 2015
D-MATH
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Applied Stochastic Processes

Exercise Sheet 4

Please hand in by 12:00 on Tuesday 24.03.2015 in the assistant's box in front of HG E 65.1

Exercise 4.1

Waiting queue paradox. Let $(N_t)_{t\geqslant 0}$ be a homogenous Poisson process with intensity λ. We define the following random variables (with $S_0 = 0$):

$$\begin{aligned} \gamma_t &= S_{N_t+1} - t \quad \text{(Waiting time for the next jump)} \\ \delta_t &= t - S_{N_t} \quad \text{(Passed time since the last jump or the point in time 0)} \\ \beta_t &= \gamma_t + \delta_t \end{aligned}$$

(a) Determine the distribution of the random variables γ_t and δ_t .

(b) Determine the joint distribution of γ_t and δ_t and explain the result.

(c) Compute the distribution of the random variable β_t.

(d) Compute $E[\beta_t]$ and $\lim_{t\to\infty} E[\beta_t]$. Compare the expectation of β_t with the interarrival times T_i. Give an interpretation of your results.

Exercise 4.2

Let $(N_t)_{t\geqslant 0}$ be an inhomogeneous Poisson process with $N_0 = 0$ and rate $\rho(t) = \alpha t$, where α is a positive constant. Let

$$W_n := \inf\{t > 0 \; : \; N_t = n\}, \quad n = 1, 2, \ldots$$

Determine the distribution of W_n and the joint distribution of $W_1, W_2 - W_1$.

Exercise 4.3

Largest gap in a Poisson process. Let $(N_t)_{t\geq 0}$ be a homogeneous Poisson process with parameter $\lambda > 0$. The largest gap up to time t is defined as

$$L_t = \max_{k\geq 1}(S_k \wedge t - S_{k-1} \wedge t).$$

In this exercise we are going to show that $\mathbb{P}$-almost surely

$$\limsup_{t\to\infty} \frac{L_t}{\log t} \leq \lambda^{-1}.$$

(a) Let $\varepsilon > 0$. Use Borel-Cantelli's lemma to show that $\mathbb{P}$-almost surely

$$\max_{1\leq k\leq n} T_k \leq \frac{1+\varepsilon}{\lambda} \log(n/\lambda)$$

for n large enough, where the T_k denote the inter-arrival times of the process.

(b) Show that $\mathbb{P}$-almost surely

$$N_t + 1 \leq (1+\varepsilon)t\lambda$$

for large t enough.

(c) Conclude that $\mathbb{P}$-almost surely $\limsup_{t\to\infty} \frac{L_t}{\log t} \leq \lambda^{-1}$.