ETH Zürich FS 2015
D-MATH
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Applied Stochastic Processes

Solution Sheet 1

Solution 1.1

(a) We assumed that the moment generating function of X exists and is finite on an open neighbourhood I of 0. This implies that X is finite outside of a $\mathbb{P}$-null set.

Let $t \in I$. Then, there exists a positive ϵ such that the open ball with center t and radius 3ϵ and the open ball with center 0 and radius 2ϵ are contained in I.

We first want to prove that $(t, \omega) \mapsto X(\omega)\mathrm{e}^{tX(\omega)}$ is integrable on $[t-\epsilon, t+\epsilon] \times \Omega$. By Fubini-Tonelli's theorem, it suffices to show that the successive integrations yield a finite value.

We prove that for $t \in I$, $X\mathrm{e}^{tX}$ is integrable, and without loss of generality, we assume that $t \geqslant 0$ (the case where $t \leqslant 0$ is treated similarly). We have,

$$\begin{aligned}
\left|X\mathrm{e}^{tX}\right| &\leqslant \left(X^{+}+X^{-}\right)\mathrm{e}^{tX^{+}} \\
&\leqslant X^{+}\mathrm{e}^{tX^{+}} + X^{-} \\
&\leqslant \frac{1}{\epsilon}\mathrm{e}^{(t+\epsilon)X^{+}} + \frac{1}{\epsilon}\mathrm{e}^{\epsilon X^{-}} \\
&\leqslant \frac{1}{\epsilon}\mathrm{e}^{(t+\epsilon)X} + \frac{1}{\epsilon}\mathbf{1}\left(X \leqslant 0\right) + \frac{1}{\epsilon}\mathrm{e}^{-\epsilon X} + \frac{1}{\epsilon}\mathbf{1}\left(X \geqslant 0\right),
\end{aligned}$$

where we define $X^{+} = \max\{X, 0\}$ and $X^{-} = \max\{-X, 0\}$ so that $X = X^{+} - X^{-}$. We used that for $x \geqslant 0$, we have $x \leqslant \mathrm{e}^{x}$. This shows that $\left|X\mathrm{e}^{tX}\right|$ is integrable on Ω for every t in I.

We now prove that $t \mapsto \mathbb{E}\left[\left|X\mathrm{e}^{tX}\right|\right]$ is continuous on $[t-\epsilon, t+\epsilon]$. For that, we use the dominated convergence theorem. Without loss of generality, let us assume that t is strictly positive. Let $0 < \delta \leqslant \epsilon' \leqslant \epsilon$ such that $t - \epsilon' > 0$. We have

$$\left|X\mathrm{e}^{(t+\delta)X}\right| \leqslant \frac{1}{\epsilon'}\mathrm{e}^{(t+2\epsilon')X} + \frac{1}{\epsilon'}\mathbf{1}\left(X \leqslant 0\right) + \frac{1}{\epsilon'}\mathrm{e}^{-\epsilon' X} + \frac{1}{\epsilon'}\mathbf{1}\left(X \geqslant 0\right)$$

which is integrable with respect to $\mathbb{P}$ on Ω and does not depend on δ. Furthermore, it holds that,

$$\lim_{\delta\to 0}\left|X\mathrm{e}^{(t+\delta)X}\right| = \left|X\mathrm{e}^{tX}\right|, \ \mathbb{P}\text{-a.s.}$$

The dominated convergence theorem then gives the continuity of the function $t \mapsto \mathbb{E}\left[\left|X\mathrm{e}^{tX}\right|\right]$. This function is therefore integrable on $[t-\epsilon, t+\epsilon]$. This gives that $(t, \omega) \mapsto X(\omega)\mathrm{e}^{tX(\omega)}$ is integrable on $[t-\epsilon, t+\epsilon] \times \Omega$.

We can now apply Fubini's theorem to the function $s \mapsto \int_{t-\epsilon}^{s} \mathbb{E}\left[X\mathrm{e}^{uX}\right] du$. We get

$$\int_{t-\epsilon}^{s} \mathbb{E}\left[X\mathrm{e}^{uX}\right] du = \mathbb{E}\left[\int_{t-\epsilon}^{s} X\mathrm{e}^{uX} du\right] = \mathbb{E}\left[\mathrm{e}^{sX}\right] - \mathbb{E}\left[\mathrm{e}^{(t-\epsilon)X}\right], \ \mathbb{P}\text{-a.s.}$$

By a similar argument as the one above, one can easily prove that $t \mapsto \mathbb{E}\left[X\mathrm{e}^{tX}\right]$ is continuous on $[t-\epsilon, t+\epsilon]$. Therefore, the derivative with respect to s of the left-hand side is the term inside the integral. This yields the result: we have proved that f is differentiable on I, with derivative $f'(t) = \mathbb{E}\left[X\mathrm{e}^{tX}\right]$.

This argument can be reproduced to prove that f is n times differentiable for all integers n, with n-th derivative $f^{(n)}(t) = \mathbb{E}\left[X^n\mathrm{e}^{tX}\right]$.

The derivative of F at 0 is then

$$\begin{aligned} F'(0) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\log\left(\mathbb{E}\left[\mathrm{e}^{tX}\right]\right)\right)\Big|_{t=0} \\ &= \frac{\mathbb{E}\left[X\mathrm{e}^{tX}\right]}{\mathbb{E}\left[\mathrm{e}^{tX}\right]}\Big|_{t=0} \\ &= \mathbb{E}\left[X\right]. \end{aligned}$$

Differentiating a second time with respect to t yields

$$\begin{aligned} F''(0) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbb{E}\left[X\mathrm{e}^{tX}\right]}{\mathbb{E}\left[\mathrm{e}^{tX}\right]}\right)\Bigg|_{t=0} \\ &= \left(\frac{\mathbb{E}\left[X^2\mathrm{e}^{tX}\right]}{\mathbb{E}\left[\mathrm{e}^{tX}\right]} - \frac{\mathbb{E}\left[X\mathrm{e}^{tX}\right]^2}{\mathbb{E}\left[\mathrm{e}^{tX}\right]^2}\right)\Bigg|_{t=0} \\ &= \mathbb{E}\left[X^2\right] - \mathbb{E}\left[X\right]^2 \\ &= \mathrm{Var}(X). \end{aligned}$$

(b) For a Poisson random variable with parameter λ and $t \in \mathbb{R}$, we have

$$\begin{aligned} \mathbb{E}\left[\mathrm{e}^{tX}\right] &= \sum_{n=0}^{\infty} \mathrm{e}^{tn}\mathbb{P}\left[X = n\right] \\ &= \sum_{n=0}^{\infty} \mathrm{e}^{tn}\frac{\lambda^n}{n!}\mathrm{e}^{-\lambda} \\ &= \mathrm{e}^{-\lambda}\sum_{n=0}^{\infty} \frac{\left(\mathrm{e}^t\lambda\right)^n}{n!} \\ &= \mathrm{e}^{\lambda\left(\mathrm{e}^t-1\right)}, \end{aligned}$$

and so $F(t) = \lambda\left(\mathrm{e}^t - 1\right)$ for $t \in \mathbb{R}$. Differentiating once gives $F'(t) = \lambda\mathrm{e}^t$ for $t \in \mathbb{R}$, and then for all $n \geqslant 2$ we have $F^{(n)}(t) = \lambda\mathrm{e}^t$. Therefore for all $n \in \mathbb{N}$, $c_n = \lambda$.

Solution 1.2

The density of the Gamma(k, λ) distribution is $f_{\Gamma(k,\lambda)}(t) = \lambda^k\frac{t^{k-1}}{(k-1)!}\mathrm{e}^{-\lambda t}$ for $t > 0$. We prove by induction that this is the density of S_k for all natural integers.

Basis: First, $S_1 = T_1$ is exponentially distributed with parameter λ with density $g(t) = \lambda\mathrm{e}^{-\lambda t}$, so S_1 has a Gamma$(1, \lambda)$ distribution.

Induction step: Assume S_k is Gamma(k,λ)-distributed and calculate the density for S_{k+1} (using independence of T_{k+1} and S_k, and convolution):

$$g^{*(k+1)}(t) = g * g^{*k}(t) = \int_0^t \lambda \mathrm{e}^{-\lambda(t-s)} \lambda^k \frac{s^{k-1}}{(k-1)!} \mathrm{e}^{-\lambda s} ds$$
$$= \lambda^{k+1} \mathrm{e}^{-\lambda t} \int_0^t \frac{s^{k-1}}{(k-1)!} ds = \lambda^{k+1} \frac{t^k}{k!} \mathrm{e}^{-\lambda t}$$

Hence S_{k+1} is Gamma$(k+1,\lambda)$-distributed and the induction is complete.

Remarks:

- Gamma(ν,λ) distribution is defined for general parameters $\nu,\lambda > 0$ and has density

$$\lambda^\nu \frac{t^{\nu-1}}{\Gamma(\nu)} \mathrm{e}^{-\lambda t}, \quad t > 0,$$

  where $\Gamma(\nu) = \displaystyle\int_0^\infty t^{\nu-1} \mathrm{e}^{-t} dt$ is the gamma function.

- For $\nu = k \in \mathbb{N}$ this is also called the Erlang-k distribution.
- We can calculate the characteristic function of the Gamma(k,λ) distribution via the characteristic function of Exp(λ):

$$\varphi_{T_1}(u) = \mathbb{E}\left[\mathrm{e}^{\mathrm{i}uT_1}\right] = \int_0^\infty \lambda \mathrm{e}^{-(\lambda-\mathrm{i}u)t} dt = \frac{\lambda}{\lambda - \mathrm{i}u},$$
$$\varphi_{S_k}(u) = \mathbb{E}\left[\mathrm{e}^{\mathrm{i}u\sum_{j=1}^k T_j}\right] = \mathbb{E}\left[\prod_{j=1}^k \mathrm{e}^{\mathrm{i}uT_j}\right] \overset{\text{iid}}{=} \mathbb{E}\left[\mathrm{e}^{\mathrm{i}uT_1}\right]^k = \left(\frac{\lambda}{\lambda - \mathrm{i}u}\right)^k.$$

Solution 1.3

(a) For all $r > 0$ we have

$$\{D > r\} \subset \{N(B_r) = 0\} \subset \{D \geq r\} \tag{1}$$

Thus, we have on the one hand

$$\mathbb{P}[D > r] \leq \mathbb{P}[N(B_r) = 0] = \mathrm{e}^{-\lambda\pi r^2} \tag{2}$$

and on the other hand

$$\mathbb{P}[D > r] = \lim_{n\to\infty} \mathbb{P}[D \geq r + 1/n] \geq \limsup_{n\to\infty} \mathbb{P}[N(B_{r+1/n}) = 0]$$
$$= \limsup_{n\to\infty} \mathrm{e}^{-\lambda\pi(r+1/n)^2} = \mathrm{e}^{-\lambda\pi r^2}, \tag{3}$$

yielding $\mathbb{P}[D > r] = \mathrm{e}^{-\lambda\pi r^2}$. Hence, the distribution function F and density f of D are given by

$$F(r) = 1 - \mathrm{e}^{-\lambda\pi r^2} \quad \text{and} \quad f(r) = 2\lambda\pi r \mathrm{e}^{-\lambda\pi r^2}, \quad r > 0. \tag{4}$$

(b) Note that $B_R \backslash B_r$ and B_r are disjoint sets, whence $N(B_R \backslash B_r)$ and $N(B_r)$ are independent. Hence, we have

$$f(R, r) = \mathbb{P}\left[N(B_R \setminus B_r) = 0 \mid N(B_r) = 1\right] = \mathbb{P}\left[N(B_R \setminus B_r) = 0\right] = \mathrm{e}^{-\lambda\pi(R^2 - r^2)}. \quad (5)$$

This immediately implies that

$$\lim_{R \searrow 0} \lim_{r \searrow 0} f(R, r) = \lim_{r \searrow 0} \lim_{R \searrow r} f(R, r) = \mathrm{e}^0 = 1. \quad (6)$$

Intuitively, $f(R, r)$ is the probability that no point lies in the annulus $B_R \setminus B_r$ given that 1 point lies in the small circle B_r. As the number of points in disjoints sets are independent, the conditioning doesn't matter. Moreover, the expected number of points in each set is equal to λ times its area. Hence as the area shrinks to 0, the expected number of points in the area goes to 0 and the probability that no point lies in the area goes to 1.