ETH Zürich FS 2015
D-MATH
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Applied Stochastic Processes

Solution Sheet 2

Solution 2.1

(a) Let A be a bounded Borel set. Notice that for some $n_0 \in \mathbb{N}$,

$$A \subset [0, a_n], \ \forall n \geqslant n_0.$$

Let us compute the characteristic function of $N_n(A)$ for $n \geqslant n_0$ at $t \in \mathbb{R}$:

$$\begin{aligned}
\phi_{N_n(A)}(t) &= \mathbb{E}\left[\mathrm{e}^{\mathrm{i}tN_n(A)}\right] \\
&= \prod_{i=1}^{n} \mathbb{E}\left[\mathrm{e}^{\mathrm{i}t1(S_i \in A)}\right] \\
&= \mathbb{E}\left[\mathrm{e}^{\mathrm{i}t1(S_1 \in A)}\right]^n \\
&= \left(\left(1 - \frac{|A|}{a_n}\right) + \frac{|A|}{a_n}\mathrm{e}^{\mathrm{i}t}\right)^n \\
&= \left(1 + \frac{\lambda|A|}{n}\left(\mathrm{e}^{\mathrm{i}t} - 1\right)\right)^n \\
&\xrightarrow{n\to\infty} \exp\left(\lambda|A|\left(\mathrm{e}^{\mathrm{i}t} - 1\right)\right)
\end{aligned}$$

The function $t \mapsto \exp\left(\lambda|A|\left(\mathrm{e}^{\mathrm{i}t} - 1\right)\right)$ is continuous at 0, so the sequence $(N_n(A))_{n\in\mathbb{N}}$ converges in distribution towards a Poisson distributed random variable with parameter $\lambda|A|$.

(b) Let $(t_1, \ldots, t_k) \in \mathbb{R}^k$, and $A_1, A_2, \ldots, A_k$ be Borel sets such that for $n \geqslant n_0$, we have $A_1, A_2, \ldots, A_k \subset [0, a_n]$. We compute the characteristic function of the random vector $(N^n(A_1), N^n(A_2), \ldots, N^n(A_k))$ at point $(t_1, \ldots, t_k) \in \mathbb{R}^k$:

$$
\begin{aligned}
\phi_{(N^n(A_1),N^n(A_2),\dots,N^n(A_k))}(t_1,\dots,t_k) &= \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\sum_{j=1}^k t_j N_n(A_j)}\right] \\
&= \mathbb{E}\left[\exp\left(\mathrm{i}\sum_{j=1}^k t_j \sum_{l=1}^n 1\left(S_l \in A_j\right)\right)\right] \\
&= \mathbb{E}\left[\exp\left(\mathrm{i}\sum_{l=1}^n \sum_{j=1}^k t_j 1\left(S_l \in A_j\right)\right)\right] \\
&= \prod_{l=1}^n \mathbb{E}\left[\exp\left(\mathrm{i}\sum_{j=1}^k t_j 1\left(S_l \in A_j\right)\right)\right] \\
&= \mathbb{E}\left[\exp\left(\mathrm{i}\sum_{j=1}^k t_j 1\left(X_1 \in A_j\right)\right)\right]^n \\
&= \left(\left(1-\frac{\lambda}{n}\sum_{j=1}^k |A_j|\right)+\sum_{j=1}^k \mathrm{e}^{\mathrm{i}t_j}\frac{\lambda|A_j|}{n}\right)^n
\end{aligned}
$$

$$
\begin{aligned}
&\xrightarrow{n\to\infty} \exp\left(\lambda\left(\sum_{j=1}^k |A_j|\left(\mathrm{e}^{\mathrm{i}t_j}-1\right)\right)\right) \\
&= \prod_{i=1}^k \exp\left(\lambda\left(|A_j|\left(\mathrm{e}^{\mathrm{i}t_j}-1\right)\right)\right),
\end{aligned}
$$

where we used for the fourth equality that the X_i's are independent, for the fifth equality, that the X_i's are identically distributed, and for the sixth equality, that the X_i's are uniformly distributed in $[0, \frac{n}{\lambda}]$. The sequence of characteristic functions of the vectors $(N^n(A_1), N^n(A_2), \dots, N^n(A_k))$ converges pointwise towards the product of characteristic functions of Poisson random variables with parameter $\lambda|A_1|, \lambda|A_2|, \dots, \lambda|A_k|$. The limit function is continuous at 0, so the sequence of random vectors converges to a vector of independent Poisson-distributed random variables with parameters $\lambda|A_1|, \lambda|A_2|, \dots, \lambda|A_k|$.

(c) By the previous question, for $0 \leqslant t_1 < t_2 < \cdots < t_k < \infty$, the sequence of vectors $\left(\left(N^n_{t_1}, N^n_{t_2} - N^n_{t_1}, N^n_{t_3} - N^n_{t_2}, \dots, N^n_{t_k} - N^n_{t_{k-1}}\right)\right)_{n\in\mathbb{N}}$ converges in distribution towards $\left(\left(N_{t_1}, N_{t_2} - N_{t_1}, N_{t_3} - N_{t_2}, \dots, N_{t_k} - N_{t_{k-1}}\right)\right)$ where N is a Poisson process with rate λ. The map that transforms $(x_1, x_2, \dots, x_k)$ into $\left(x_1, x_2 + x_1, x_3 + x_2 + x_1, \dots, \sum_{j=1}^k x_j\right)$ is a continuous bijection, therefore the sequence of vectors $\left(\left(N^n_{t_1}, N^n_{t_2}, N^n_{t_3}, \dots, N^n_{t_k}\right)\right)_{n\in\mathbb{N}}$ converges in distribution to $(N_{t_1}, N_{t_2}, N_{t_3}, \dots, N_{t_k})$. The processes N^n converge to a Poisson process with rate λ in the sense of finite-dimentional distributions.

Solution 2.2

For $n \in \mathbb{N}$ set $\widetilde{T}_n := -\log(U_n)/\lambda$. Clearly, the $\widetilde{T}_n$ are i.i.d. as the U_n and we have for $t \geq 0$

$$
M_t = \sup\left\{n \in \mathbb{N}_0 : \sum_{k=1}^n \widetilde{T}_k \leq t\right\}. \tag{1}
$$

Moreover, the $\widetilde{T}_n$ are exponentially distributed. Indeed, let $x \in \mathbb{R}$. Then we have

$$\mathbb{P}[\widetilde{T}_1 \le x] = \mathbb{P}[\log U_1 \ge -\lambda x] = \mathbb{P}[U_1 \ge \mathrm{e}^{-\lambda x}] = \begin{cases} 0 & \text{if } x \le 0, \\ 1 - \mathrm{e}^{-\lambda x} & \text{if } x > 0. \end{cases} \tag{2}$$

a) First, note that for $0 \le s < t$ we have $\{M_s = +\infty\} \subset \{M_t = +\infty\}$. Hence, we have $\bigcup_{t \ge 0}\{M_t = +\infty\} = \bigcup_{j \in \mathbb{N}}\{M_j = +\infty\}$. To establish the first claim, it therefore suffices to show that for all $j \in \mathbb{N}$ we have $\mathbb{P}[M_j = +\infty] = 0$. Fix $j \in \mathbb{N}$. Using the independence of the $\widetilde{T}_n$ we have

$$\begin{aligned} \mathbb{P}[M_j = +\infty] \le \mathbb{P}[\bigcap_{k \in \mathbb{N}}\{\widetilde{T}_k \le j\}] &= \prod_{k=1}^{\infty} \mathbb{P}[\widetilde{T}_k \le j] \\ &= \prod_{k=1}^{\infty}(1 - \mathrm{e}^{-\lambda j}) = 0. \end{aligned} \tag{3}$$

Next, it follows immediately from the definition that (N_t) starts at 0 and has nondecreasing trajectories with values in $\mathbb{N}_0$. It remains to show that the sample paths are right-continuous. Clearly we have to check this property only outside the set $\bigcup_{t \ge 0}\{M_t = +\infty\}$. Fix $t \ge 0$ and let $\omega \in \bigcap_{t \ge 0}\{M_t < \infty\}$ and $n := N_t(\omega) = M_t(\omega) \in \mathbb{N}_0$. Then we have by definition of M_t

$$\sum_{k=1}^{n} \widetilde{T}_k(\omega) \le t \quad \text{and} \quad \sum_{k=1}^{n+1} \widetilde{T}_k(\omega) > t. \tag{4}$$

Hence, for all $\epsilon > 0$ sufficiently small we also have

$$\sum_{k=1}^{n} \widetilde{T}_k(\omega) \le t + \epsilon \quad \text{and} \quad \sum_{k=1}^{n+1} \widetilde{T}_k(\omega) > t + \epsilon. \tag{5}$$

Therefore, for all $\epsilon > 0$ sufficiently small we have $N_{t+\epsilon}(\omega) = n = N_t(\omega)$ implying that the function $s \mapsto N_s(\omega)$ is right-continuous at $s = t$.

b) Denote by $(S_n)_{n \in \mathbb{N}}$ the sequence of jump times of N. For $\omega \in \bigcap_{t \ge 0}\{M_t < +\infty\}$, it follows immediately from the definition of M and N that $N.(\omega)$ increase by jumps of size 1 and that $S_n(\omega) = \sum_{k=1}^{n} \widetilde{T}_k(\omega) < \infty$, $n \in \mathbb{N}$. (Note that $\widetilde{T}_n(\omega) \in (0, \infty)$ for all $n \in \mathbb{N}$). For $\omega \in \bigcup_{t \ge 0}\{M_t = +\infty\}$, we have $N.(\omega) \equiv 0$ and $S_n(\omega) = +\infty$ for all $n \in \mathbb{N}$. In conclusion, N increases by jumps of size 1, and we have $S_n < \infty$ $\mathbb{P}$-a.s. for all $n \in \mathbb{N}$. Denote by $(T_n)_{n \in \mathbb{N}}$ the sequence of interarrival times of N. This is well defined on $\bigcap_{t \ge 0}\{M_t < +\infty\}$, where we have $T_n = \widetilde{T}_n$. In particular, the T_n are i.i.d. and distributed as the $\widetilde{T}_n$, i.e. $T_n \sim \mathrm{Exp}(\lambda)$. This establishes the claim as we know from the lecture that a counting process with jumps of size 1 starting at 0 and having i.i.d. interarrival times that are exponentially distributed with parameter $\lambda > 0$, is a Poisson process with rate λ.

Solution 2.3

Denote by $\mathrm{Sym}(n)$ the symmetric group of degree n. Since the X_i have a density, we have $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$ $\mathbb{P}$-a.s. Using that the X_i are i.i.d. and that the order of $\mathrm{Sym}(n)$ is $n!$, we get for all $B \in \mathcal{B}(\mathbb{R}^n)$

$$\begin{aligned}
\mathbb{P}[(X_{(1)}, \ldots, X_{(n)}) \in B] &= \mathbb{P}[(X_{(1)}, \ldots, X_{(n)}) \in B, X_{(1)} < X_{(2)} < \cdots < X_{(n)}] \\
&= \sum_{\pi \in \mathrm{Sym}(n)} \mathbb{P}[(X_{(1)}, \ldots, X_{(n)}) \in B, X_{(1)} < \cdots < X_{(n)}, X_{(1)} = X_{\pi(1)}, \ldots, X_{(n)} = X_{\pi(n)}] \\
&= \sum_{\pi \in \mathrm{Sym}(n)} \mathbb{P}[(X_{\pi(1)}, \ldots, X_{\pi(n)}) \in B, X_{\pi(1)} < X_{\pi(2)} < \cdots < X_{\pi(n)}] \\
&= n!\, \mathbb{P}[(X_1, \ldots, X_n) \in B, X_1 < X_2 < \cdots < X_n] \\
&= \int_{\mathbb{R}^n} \mathbb{1}_{\{(x_1, \ldots, x_n) \in B\}} n!\, \mathbb{1}_{\{x_1 < x_2 < \cdots < x_n\}} \prod_{i=1}^{n} f(x_i)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n. \qquad (6)
\end{aligned}$$

This establishes the claim.