ETH Zürich FS 2015
D-MATH
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Coordinator
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Applied Stochastic Processes

Exercise Sheet 2

Please hand in by 12:00 on Tuesday 10.03.2015 in the assistant's box in front of HG E 65.1

Exercise 2.1

Let $n \in \mathbb{N}$. We consider here a queuing model, where n clients are arriving at random (uniformly) during opening hours. The shop is open from time $t = 0$ to time $t = a_n > 0$. Let $S_1, S_2, \ldots, S_n$ be i.i.d. random variables that are uniformly distributed on $[0, a_n]$. Let A be a bounded Borel set, we define

$$N^n(A) = \sum_{i=1}^{n} 1\,(S_i \in A)\,.$$

We choose $a_n = \frac{n}{\lambda}$, for some constant $\lambda > 0$.

(a) Prove that $(N^n(A))_{n\in\mathbb{N}}$ converges to a Poi$(\lambda|A|)$ random variable.

(b) Let $k \in \mathbb{N}$ and $A_1, A_2, \ldots, A_k$ be disjoint and bounded Borel sets. Show that the sequence $(N^n(A_1), N^n(A_2), \ldots, N^n(A_k))_{n\in\mathbb{N}}$ converges in distribution towards a vector of independent Poisson random variables with parameters $\lambda|A_1|,\ \lambda|A_2|\ , \ldots,\ \lambda|A_k|$.

(c) Conclude that the sequence of processes defined as $N_t^n := N^n([0,t])$ for $t \geqslant 0$ and $n \in \mathbb{N}$ converges to a Poisson process in the sense of finite-dimensional distributions, i.e.

$$\left(N_{t_1}^n, N_{t_2}^n, \ldots, N_{t_k}^n\right) \xrightarrow[n\to\infty]{d} \left(N_{t_1}, N_{t_2}, \ldots, N_{t_k}\right),\ \forall\ 0 \leqslant t_1 < t_2 < \cdots < t_k < \infty,$$

where N is a Poisson process with rate λ.

This means that in the limit, if the ratio $\dfrac{\text{length of the opening time}}{\text{number of clients}}$ tends to a constant when both quantities become large, then the number of clients in a given time interval converges to a Poisson random variable with parameter: limit of the ratio×length of the time interval.

Exercise 2.2

Let $(U_k)_{k\in\mathbb{N}}$ be a sequence of i.i.d. random variables which are uniformly distributed on $(0,1)$ and $\lambda > 0$. For each $t \geq 0$ define a random variable M_t valued in $\mathbb{N}_0 \cup \{+\infty\}$ by

$$M_t := \sup\Big\{n \in \mathbb{N}_0 : -\sum_{k=1}^{n} \log U_k \leq \lambda t\Big\}.$$

a) Show that $\mathbb{P}[\bigcup_{t\geq 0}\{M_t = +\infty\}] = 0$ and that the stochastic process $(N_t)_{t\geq 0}$ defined by $N_t := M_t \mathbb{1}_{\bigcap_{t\geq 0}\{M_t<+\infty\}}$ is a counting process starting at 0.

b) Show that N is a standard Poisson process with rate λ.

The above result can be used to *simulate* a standard Poisson processes on a computer.

Exercise 2.3

Let $X_1, \dots, X_n$ be real-valued i.i.d. random variables with a density f. Denote by $X_{(1)}, \dots, X_{(n)}$ the *order statistics* of $X_1, \dots, X_n$, that we define recursively as

$$X_{(1)} := \min\{X_1, \dots, X_n\},$$
$$\text{and for } k \in \{2, \dots, n\},\ X_{(k)} := \min\{\{X_1, \dots, X_n\} \backslash \{X_{(1)}, ..., X_{(k-1)}\}\}.$$

Equivalently, $X_{(1)}, \dots, X_{(n)}$ is defined as $X_{\pi(1)}, ..., X_{\pi(n)}$, where π is a permutation (depending on the X_i's) such that $X_{\pi(1)} < X_{\pi(2)} < ... < X_{\pi(n)}$.

Show that the joint density g of $X_{(1)}, \dots, X_{(n)}$ is given by

$$g(x_1, \dots, x_n) = n! \prod_{i=1}^{n} f(x_i) \mathbb{1}_{\{x_1 < x_2 < \cdots < x_n\}}.$$