ETH Zürich FS 2015
D-MATH
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Coordinator
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Applied Stochastic Processes

Exercise Sheet 7

Please hand in by 12:00 on Tuesday 21.04.2015 in the assistant's box in front of HG E 65.1

Exercise 7.1

A die is rolled repeatedly. Which of the following stochastic processes $(X_n)_{n\in\mathbb{N}}$ are Markov chains? For those that are, determine the transition matrix and in b), additionally, the n-step transition matrix.

(a) Let X_n denote the number of rolls at time n since the most recent six.

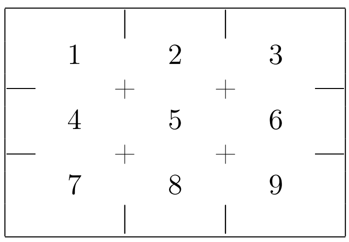
(b) Let X_n denote the largest number that has come up in the first n rolls.

(c) Let X_n denote the larger number of those that came up in the rolls number $n-1$ and n (the last two rolls), and we consider $(X_n)_{n\geq 2}$.

Exercise 7.2

Determine the transition matrices for the following homogeneous Markov chains $(X_n)_{n\in\mathbb{N}}$:

(a) A rat moves randomly in the maze shown by the figure below.



When it leaves a room, it visits one of the neighbouring rooms with equal probabilities. Denote by $(X_n)_{n\in\mathbb{N}}$ the sequence of rooms that the rat visits.

(b) N black and N red balls are placed in two urns so that each urn contains N balls. In each step a ball is drawn at random from each urn, and each of the two balls is put into the other urn so that each urn always contains N balls. Denote by X_n, $n \in \mathbb{N}$, the number of red balls in the first urn after n steps.

(c) A coin is tossed repeatedly with $\mathbb{P}[\text{"head"}] = p \in (0,1)$. Denote by Y_n, $n \in \mathbb{N}$, the outcome of the n-th coin toss, where we interpret 1 as "head" and 0 as "tails". Fix $k \in \mathbb{N}$ and define $X_n := (Y_{n+1}, Y_{n+2}, \dots, Y_{n+k})$.

Hint: You can identify X_n with the corresponding binary number $\sum_{i=1}^{k} Y_{n+i}2^{k-i}$.

Exercise 7.3

Inhomogeneous Markov chains

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(X_n)_{n\in\mathbb{N}}$ a sequence of random variables valued in some nonempty, at most countable set E endowed with the σ-algebra $\mathcal{E} := 2^E$.

For $n \in \mathbb{N}$ define the linear operator $R(n)$ from the set of bounded functions on E in itself by

$$(R(n)f)(x) := \begin{cases} \mathbb{E}\left[f(X_n) \mid X_{n-1} = x\right], & \text{for } x \in E \text{ if } \mathbb{P}[X_{n-1} = x] > 0, \\ f(x) & \text{for } x \in E \text{ if } \mathbb{P}[X_{n-1} = x] = 0, \end{cases}$$

for $f \in L^\infty(E)$.

(a) Show that this linear operator is bounded, with $||R(n)|| \leqslant 1$ for all integer n.

One can identify this bounded linear operator with the (possibly infinite) matrix $R(n) \in [0,1]^{E\times E}$ defined as

$$R_{x,y}(n) := \begin{cases} (R(n)\delta_y)(x) = \mathbb{P}[X_n = y \,|\, X_{n-1} = x] & \text{if } \mathbb{P}[X_{n-1} = x] > 0, \\ \delta_{x,y} & \text{if } \mathbb{P}[X_{n-1} = x] = 0, \end{cases}$$

where δ denotes *Kronecker's delta.* We identify any function $f : E \to \mathbb{R}$ with the (possibly infinite) column vector $f \in \mathbb{R}^E$ defined by $f_x := f(x)$. Likewise, we identify any probability measure μ on E with the (possibly infinite) row vector $\mu \in [0,1]^E$ defined by $\mu_x = \mu(\{x\})$.

(b) Show that $(X_n)_{n\in\mathbb{N}}$ is a *discrete-time Markov chain* if and only if for all $n \in \mathbb{N}$ and all bounded functions $f : E \to \mathbb{R}$ we have

$$\mathbb{E}[f(X_{n+1}) \,|\, X_0, \ldots, X_n] = (R(n+1)f)_{X_n} \;\; \mathbb{P}\text{-a.s.}$$

(c) Let μ be any distribution on E. Show that $(X_n)_{n\in\mathbb{N}}$ is a discrete-time Markov chain with $X_0 \sim \mu$ under $\mathbb{P}$ if and only if for all $n \in \mathbb{N}$ and all $x_0, \ldots, x_n \in E$ we have

$$\mathbb{P}[X_0 = x_0, \ldots, X_n = x_n] = \mu_{x_0} R_{x_0,x_1}(1) R_{x_1,x_2}(2) \times \cdots \times R_{x_{n-1},x_n}(n).$$

(d) Suppose that $(X_n)_{n\in\mathbb{N}}$ is a discrete-time Markov chain such that $X_0 \sim \mu$ under $\mathbb{P}$. Let $f : E \to \mathbb{R}$ be a bounded function. Show that

$$\mathbb{E}[f(X_n)] = \mu R(1) R(2) \cdots R(n) f, \quad n \geq 0.$$

(e) Suppose that $(X_n)_{n\in\mathbb{N}}$ is a discrete-time Markov chain. Show that $(X_n)_{n\in\mathbb{N}}$ is a *homogeneous* Markov chain if and only if there exists a transition matrix $R \in [0,1]^{E\times E}$ such that for all $n \in \mathbb{N}$ and all $y \in E$ we have

$$R_{x,y} = R_{x,y}(n+1) \quad \text{if} \quad \mathbb{P}[X_n = x] > 0.$$

Exercise 7.4

Let $(N_t)_{t\geqslant 0}$ be a Poisson process with rate $\lambda > 0$. Denotes its arrival times by $S_1, S_2, \ldots$ and the interarrival times by $T_1, T_2, \ldots$

Consider claims arriving at an insurance company according to N. The non-negative claim sizes X_i, $i \in \{1, 2, \ldots, \}$ are i.i.d. with common distribution function G. For $x \geqslant 0$, define the risk process $f_x(t)$, which corresponds to the capital reserves of the firm at time t, by

$$f_x(t) = x + ct - \sum_{k=1}^{N_t} X_k.$$

The constant $c > 0$ is the rate at which the firm receives the premium.

Define the no-ruin probability starting with capital x as

$$R(x) = \mathbb{P}[f_x(t) \geq 0 \text{ for all } t > 0].$$

(a) Show that if $\lambda \mathbb{E}[X_1] < c$, then R' satisfies the renewal equation with defect

$$R'(t) = F'(t)R(0) + \int_0^t R'(t-s)\,\mathrm{d}F(s), \qquad (*)$$

where $F(t) = \int_0^t \frac{\lambda}{c}\mathbb{P}[X_1 > u]\,\mathrm{d}u$ for $t \geq 0$.

(b) We keep the following assumption: $\lambda \mathbb{E}[X_1] < c$. Use the renewal equation $(*)$ to show

(i) $R(\infty) = 1$,

(ii) $R(0) = 1 - \frac{\lambda}{c}\mathbb{E}[X_1]$,

(iii) $R(t) = R(0)(1 + M(t))$,

where $M(t)$ is the defective renewal function corresponding to F.
Hint: To show (ii) and (iii), solve the Laplace transform version of the renewal equation $(*)$.

(c) Assume that the claim size has a second moment. The function R satisfies the renewal equation with defect $R = h + R * F$ for all $t \geqslant 0$, with $h = R(0)$. Assume that there exist an $\alpha > 0$ such that

$$\frac{\lambda}{c}\int_0^\infty \mathrm{e}^{\alpha t}\mathbb{P}\left[X_1 > t\right] dt = 1$$

Use Smith's theorem in the case of a renewal process with defect to prove

$$1 - R(t) \sim_{t\to\infty} \frac{\mathrm{e}^{-\alpha t}\left(c - \lambda \mathbb{E}\left[X_1\right]\right)}{\alpha\lambda \int_0^\infty x\mathrm{e}^{\alpha x}\mathbb{P}\left[X_1 > x\right] dx}.$$

In other words the no-ruin probability converges exponentially to 1, as the initial capital goes to infinity.