ETH Zürich FS 2015
D-MATH
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Coordinator
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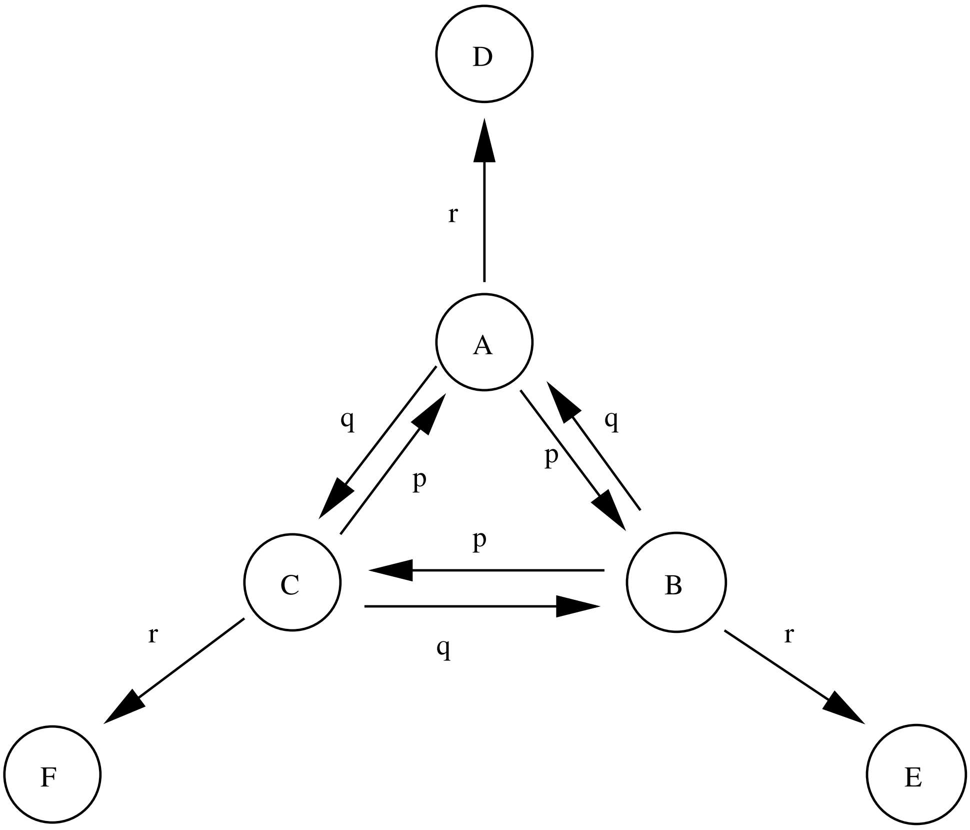
Applied Stochastic Processes

Exercise Sheet 8

Please hand in by 12:00 on Tuesday 28.04.2015 in the assistant's box in front of HG E 65.1

Exercise 8.1

Consider a homogeneous Markov chain $(X_n)_{n\in\mathbb{N}_0}$ with state space $\{A, B, C, D, E, F\}$, where the transition probabilities are illustrated by the following graph:



Here we assume that $0 < p, q, r < 1$ and $p + q + r = 1$. Suppose that the chain starts in state A. For $k \in \{D, E, F\}$ compute the probability that the chain ends up in state k.

Hint: Use the *Markov property* and the symmetry of the graph.

Exercise 8.2

Let $(X_n)_{n\geq 0}$ be a homogeneous Markov chain with countable state space E and transition probabilities $(r_{x,y})_{x,y\in E}$. Let $C \subseteq E$ such that $E\backslash C$ is finite. Define $r_{x,C} = \sum_{y\in C} r_{x,y}$. Suppose that for each $x \in E\backslash C$ there exists an $n(x)$ such that $r_{x,C}(n(x)) > 0$. Let $\tau_C = \inf\{n \geq 0 \ : \ X_n \in C\}$,

$\varepsilon = \min\{r_{x,C}(n(x)) \, : \, x \in E\backslash C\}$, and $N = \max\{n(x) \, : \, x \in E\backslash C\}$. Show that for all $k \in \mathbb{N}_0$,

$$P_x[\tau_C > kN] \leq (1-\varepsilon)^k \quad \forall x \in E.$$

Exercise 8.3

We use the same notation as in Exercise 8.2. Let $A, B \subseteq E$ with $A \cap B = \emptyset$. Suppose that $E\backslash(A \cup B)$ is finite and $P_x[\tau_{A\cup B} < \infty] > 0$ for all $x \in E\backslash(A \cup B)$.

(a) Define $h(x) = P_x[\tau_A < \tau_B]$. Prove that

$$h(x) = \sum_{y \in E} r_{x,y} h(y) \quad \text{for all } x \in E\backslash(A \cup B). \tag{$*$}$$

(b) Using Exercise 8.2, show that $P_x[\tau_{A\cup B} < \infty] = 1$.

(c) Show that if a function h on E satisfies $(*)$, then

$$E_\mu[h(X_{n\wedge\tau_{A\cup B}}) \mid \mathcal{F}_{n-1}] = h(X_{(n-1)\wedge\tau_{A\cup B}}),$$

hence $(h(X_{n\wedge\tau_{A\cup B}}))_{n\geq 0}$ is a martingale.
Optional: Use this to show that $h(x) = P_x(\tau_A < \tau_B)$ is the only solution of $(*)$ that is 1 on A and 0 on B.

Exercise 8.4

The idea of the exercise is to use Fourier transforms to prove necessary and sufficient conditions on the transition probabilities of a random walks for it to be transient or recurrent.

Let us consider a random walk X on $\mathbb{Z}$, starting at 0, with probability p of going forward and probability $q = 1-p$ to go backward. This is a discrete Markov chain, and following the notation of Exercise 7.3, we have

$$R = (r_{x,y})_{x,y\in E},$$

where the bounded linear operator R is the one defined in Exercise 7.3. Let $\xi \in [-\pi, \pi)$. We define the function e_ξ by

$$e_\xi(x) := \mathrm{e}^{\mathrm{i}x\xi}.$$

(a) Compute Re_ξ and $R^n e_\xi$.

(b) Compute

$$\int_{[-\pi,\pi)} \frac{d\xi}{2\pi} \left(R^n e_\xi\right)(0)$$

and show that $r_{0,0}(n) = \int_{[-\pi,\pi)} \frac{d\xi}{2\pi} \left(p\mathrm{e}^{\mathrm{i}x\xi} + q\mathrm{e}^{-\mathrm{i}x\xi}\right)^n$.

(c) Let $\varepsilon > 0$, and set $p = \frac{1}{2} + \frac{a}{2}$, $q = \frac{1}{2} - \frac{a}{2}$, for $a \in (-1, 1)$. We define

$$K_\varepsilon = \sum_{n\geqslant 0} \mathrm{e}^{-\varepsilon n} r_{0,0}(n).$$

Compute K_ε, and determine whether the random walk is recurent or transient on $\mathbb{Z}$. Be careful: this depends on a.

(d) Denote by e_i the canonical orthonormal basis vector of $\mathbb{Z}^d$. Extend the previous results to a random walk in $\mathbb{Z}^d$, $d \geqslant 2$, for which the transition probabilities are such that

$$\mathbb{P}\left[X_{n+1} = x + e_i \mid X_n = x\right] = \mathbb{P}\left[X_{n+1} = x - e_i \mid X_n = x\right] = \frac{b_i}{2},$$

for some $b_i > 0$ for $i \in \{1, ..., d\}$, with $\sum_{i=1}^d b_i = 1$.

This is a random walk without drift, i.e. $\mathbb{E}\left[X_n\right] = 0$ for all n, but the probability of taking a step in the dimension i need not be the same for all dimensions $i = 1, ..., d$.

Hint: consider separately $d = 2$ *and* $d \geqslant 3$.

(e) **(Optional)** Consider a general random walk on $\mathbb{Z}^d$ with arbitrary transition probabilities

$$\mathbb{P}\left[X_{n+1} = x + e_i \mid X_n = x\right] = p_i, \quad \mathbb{P}\left[X_{n+1} = x - e_i \mid X_n = x\right] = q_i,$$

for some $p_i, q_i > 0$ for $i \in \{1, ..., d\}$, with $\sum_{i=1}^d (p_i + q_i) = 1$.

Find necessary and sufficient conditions on the p_i's and the q_i's for the random walk to be transient (resp. recurrent).