ETH Zürich FS 2015
D-MATH
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Applied Stochastic Processes

Exercise Sheet 10

Please hand in by 12:00 on Tuesday 19.05.2015 in the assistant's box in front of HG E 65.1

Exercise 10.1

Consider the reflected random walk on $E = \mathbb{N}_0$, i.e. a Markov chain $(X_n)_{n\in\mathbb{N}_0}$ with transition matrix

$$R = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ q & 0 & p & 0 & \ddots \\ 0 & q & 0 & p & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

where $p, q > 0$ and $p + q = 1$.

(a) Show that $(X_n)_{n\in\mathbb{N}_0}$ is *transient*, i.e. all states $x \in E$ are transient, if $p > 1/2$.

(b) Set $\alpha := \rho_{1,0}$. Show that α satisfies the equation

$$\alpha = q + p\alpha^2.$$

Deduce that $(X_n)_{n\in\mathbb{N}_0}$ is *recurrent*, i.e. all states $x \in E$ are recurrent, if $p \leq 1/2$.

(c) Set $\beta := \mathbb{E}_1[\widetilde{H}_0]$. Show that if $(X_n)_{n\in\mathbb{N}_0}$ is *positive recurrent*, i.e. all states $x \in E$ are positive recurrent, then β satisfies the equation

$$\beta = 1 + 2p\beta.$$

Deduce that $(X_n)_{n\in\mathbb{N}_0}$ is *positive recurrent* if $0 < p < 1/2$ and *null recurrent*, i.e. all states $x \in E$ are null recurrent, if $p = 1/2$.

(d) When does the reflected random walk have a stationary distribution? Is it unique? If yes, find it.

Exercise 10.2

Metropolis-Hastings algorithm. Let π be a probability distribution on a countable state space E. Suppose we have a transition probability $(r_{x,y})_{x,y\in E}$ on E.

(a) Construct a transition probability $(r'_{x,y})_{x,y\in E}$ on E such that a transition from x to y is proposed with probability $r_{x,y}$, but only accepted with probability

$$\left(\frac{\pi(y)r_{y,x}}{\pi(x)r_{x,y}} \wedge 1\right).$$

If the transition is refused, then the chain remains at x.
(If $\pi(x)r_{x,y} = 0$, we set $\left(\frac{\pi(y)r_{y,x}}{\pi(x)r_{x,y}} \wedge 1\right) = 1$.)

(b) Show that π is a reversible distribution of the Markov chain with transition probability $(r'_{x,y})_{x,y\in E}$.

Exercise 10.3

Birth and Death Process. We recall the settings of Exercise 9.2.

Let $(X_n)_{n\in\mathbb{N}_0}$ be a Markov chain on a countable state space E, with transition probability $(r_{i,j})_{i,j\in E}$ and initial distribution μ. $(X_n)_{n\in\mathbb{N}_0}$ is called *reversible* if for all $m \in \mathbb{N}$ and all $i_0, \dots, i_m \in E$ we have

$$\mathbb{P}_\mu[X_0 = i_0, X_1 = i_1, \dots, X_m = i_m] = \mathbb{P}_\mu[X_m = i_0, X_{m-1} = i_1, \dots, X_0 = i_m].$$

Consider a birth and death process $(X_n)_{n\in\mathbb{N}}$ with state space $\mathbb{N}_0$ and transition probabilities $r_{i,j}$ with the restriction that $r_{i,j} = 0$ if $|i - j| > 1$.

(a) We define a probability distribution ν on $\mathbb{N}_0$ by

$$\nu_i := \prod_{k=1}^{i} \frac{r_{k-1,\,k}}{r_{k,\,k-1}}$$

Prove that $(\nu_i)_{i\in\mathbb{N}_0}$ is a stationary distribution.

(b) Is this Markov chain reversible?