ETH Zürich FS 2015
D-MATH
Prof. Antti Knowles

Coordinator
Thomas Cayé

Applied Stochastic Processes

Exercise Sheet 11

Please hand in by 12:00 on Tuesday 26.05.2015 in the assistant's box in front of HG E 65.1

Exercise 11.1

Consider a Markov chain in continuous time on a countable state space E. Prove the following: If one of the conditions **(a)**, **(b)** or **(c)** holds, then there is no explosion in finite time.

(a) There exists a constant $c > 0$, such that $\lambda(x) < c$ for all $x \in E$.

(b) E is finite.

(c) Let $\mathcal{T}$ be the set of all transient states of the discrete time Markov chain on the same state space and with transition matrix $(q_{x,y})_{x,y \in E}$. Suppose that $\mathbb{P}'_x[\cap_{n \geqslant 0}\{X'_n \in \mathcal{T}\}] = 0$ for all $x \in E$ and that $\lambda(x) < \infty$ for all $x \in E$.

Give an example of $(\lambda(x))_{x \in E}$ and $(q_{x,y})_{x,y \in E}$ for which the non-explosion condition is false.

Exercise 11.2

Let $(X_t)_{t \geq 0}$ denote a homogeneous continuous-time Markov chain with generator

$$\Lambda = \frac{1}{4}\begin{pmatrix} -8 & 3 & 5 \\ 6 & -8 & 2 \\ 2 & 2 & -4 \end{pmatrix}$$

Compute the matrix $R(t)$.

Exercise 11.3

Linear Growth with Immigration. We consider a birth and death process $(X_t)_{t \geq 0}$ with $\lambda_i = \lambda i + a$ and $\mu_i = \mu i$, $\lambda, \mu, a > 0$. We define $M(t) = E[X_t]$.

(a) Show that $(X_t)_{t \geq 0}$ fulfills the non-explosion assumption.

(b) Use the forward Kolmogorov differential equations to derive the following differential equation for $M(t)$

$$M'(t) = a + (\lambda - \mu)M(t),$$

with initial condition $M(0) = i$, if $X(0) = i$.

(c) Solve the differential equation for M.

Exercise 11.4

Consider a pure jump process with state space $E = \mathbb{N}$, transition probability

$$q_{x,y} = \begin{cases} p, & \text{if } y = x+1,\ x \geq 1, \\ q, & \text{if } y = 0,\ x \geq 1, \\ 1, & \text{if } y = 1,\ x = 0, \\ 0 & \text{otherwise,} \end{cases}$$

with $p + q = 1$, $p, q > 0$, and an arbitrary jump rate function $\lambda(\cdot) : \mathbb{N} \to (0, \infty)$.

(a) Show that the discrete skeleton X'_n is an irreducible discrete time Markov chain on $\mathbb{N}$ and that all states $x \in \mathbb{N}$ are positive recurrent for $(X'_n)_{n \geqslant 0}$.

(b) Deduce from **(a)** that we have a pure jump process with no explosion for any jump rate function $\lambda(\cdot)$.

(c) Show that

$$E_0[\tilde{H}_0] = \frac{1}{\lambda(0)} + \sum_{m=1}^{\infty} \frac{1}{\lambda(m)} p^{m-1},$$

where

$$\tilde{H}_0 = \inf\{t > 0,\ \ X_t = 0 \text{ and there is } s \in (0,t) \text{ with } X_s \neq 0\}.$$

(d) Find a jump rate function $\lambda(\cdot)$ such that $E_0[\tilde{H}_0] = \infty$.

Exercise 11.5

This problem uses material that will be covered in class on 26.05, and is therefore due only on 28.05.

Birth and death process. Consider a birth and death process on $\mathbb{N}$ with birth rates $(\lambda_i)_{i \geq 0}$ and death rates $(\mu_i)_{i \geq 0}$, $\mu_0 = 0$. Assume that there is no explosion in finite time.

(a) Under which conditions does a stationary distribution exist?

(b) Compute the stationary distribution if it exists.