ETH Zürich FS 2015
D-MATH
Prof. Antti Knowles

Coordinator
Thomas Cayé

Applied Stochastic Processes

Solution Sheet 3

Solution 3.1

(a) For $n \in \mathbb{N}$ set $Y_n := \sum_{i=1}^{n} |X_i|$ and note that the $|X_i|$ are i.i.d. and independent of τ. Hence, we have by the monotone convergence theorem and independence of the X_i and τ

$$
\begin{aligned}
\mathbb{E}[|S_\tau|] &= \mathbb{E}\Big[\sum_{k=1}^{\infty} |S_k| \mathbf{1}\,(\tau = k)\Big] \le \mathbb{E}\Big[\sum_{k=1}^{\infty} Y_k \mathbf{1}\,(\tau = k)\Big] \\
&= \sum_{k=1}^{\infty} \mathbb{E}[Y_k \mathbf{1}\,(\tau = k)] = \sum_{k=1}^{\infty} \mathbb{E}[Y_k]\mathbb{E}[\mathbf{1}\,(\tau = k)] \\
&= \mathbb{E}[Y_1]\Big(\sum_{k=0}^{\infty} k\mathbb{P}[\tau = k]\Big) = \mathbb{E}[Y_1]\mathbb{E}[\tau] < \infty.
\end{aligned}
$$

Next, by independence of the X_i and τ we have a.s.

$$
\mathbb{E}[S_\tau|\tau] = \mathbb{E}[S_k]\Big|_{k=\tau} = \mu k\Big|_{k=\tau} = \mu\tau,
$$

yielding the first assertion. The second assertion follows immediately from this by the tower property of conditional expectations.

(b) For all $n \in \mathbb{N}$ we have

$$
\begin{aligned}
\mathbb{E}[(S_n)^2] &= \sum_{i,j=1}^{n} \mathbb{E}[X_i X_j] = \sum_{i=1}^{n} \mathbb{E}[X_i^2] + \sum_{i,j=1, i\neq j}^{n} \mathbb{E}[X_i X_j] \\
&= n(\sigma^2 + \mu^2) + n(n-1)\mu^2 = n\sigma^2 + n^2\mu^2.
\end{aligned}
$$

Next, since $(S_\tau)^2 \ge 0$, the conditional expectation $\mathbb{E}[(S_\tau)^2|\tau]$ is well-defined. By independence of the X_i and τ we have a.s.

$$
\mathbb{E}[(S_\tau)^2|\tau] = \mathbb{E}[(S_k)^2]\Big|_{k=\tau} = k\sigma^2 + k^2\mu^2\Big|_{k=\tau} = \sigma^2\tau + \mu^2\tau^2,
$$

establishing the first assertion. By the tower property of conditional expectations we get

$$
\mathbb{E}[(S_\tau)^2] = \mathbb{E}[\mathbb{E}[(S_\tau)^2|\tau]] = \mathbb{E}[\sigma^2\tau + \mu^2\tau^2] = \sigma^2\mathbb{E}[\tau] + \mu^2\mathbb{E}[\tau^2].
$$

Putting this together with the result from part (a), we get

$$
\begin{aligned}
\mathrm{Var}(S_\tau) &= \mathbb{E}[(S_\tau)^2] - \mathbb{E}[S_\tau]^2 = \sigma^2\mathbb{E}[\tau] + \mu^2\mathbb{E}[\tau^2] - \mu^2\mathbb{E}[\tau]^2 \\
&= \sigma^2\mathbb{E}[\tau] + \mu^2\mathrm{Var}[\tau].
\end{aligned}
$$

Solution 3.2

(a) For $k \in \mathbb{N}_0$ denote by μ^{*k} the k-fold convolution of μ, where we agree that $\mu^{*1} = \mu$ and $\mu^{*0} := \delta_0$ (the *Dirac measure* at 0). Fix $t > 0$ and $B \in \mathcal{B}(\mathbb{R})$. Using that N_t and $(X_k)_{k\in\mathbb{N}}$ are independent and $N_t \sim \mathrm{Poi}(\lambda t)$, we have

$$\begin{aligned}\mathbb{P}[Z_t \in B] &= \sum_{k=0}^{\infty} \mathbb{P}[Z_t \in B, N_t = k] = \sum_{k=0}^{\infty} \mathbb{P}\Big[\sum_{j=1}^{k} X_j \in B, N_t = k\Big] \\ &= \sum_{k=0}^{\infty} \mathbb{P}\Big[\sum_{j=1}^{k} X_j \in B\Big]\mathbb{P}[N_t = k] = \sum_{k=0}^{\infty} \mu^{*k}(B)\frac{(\lambda t)^k}{k!}\mathrm{e}^{-\lambda t}. \end{aligned} \tag{1}$$

Next, fix $t > 0$ and $u \in \mathbb{R}$. Denote by φ_X the common characteristic function of the X_i. Using that N_t and $(X_k)_{k\in\mathbb{N}}$ are independent, the X_i are i.i.d. and $N_t \sim \mathrm{Poi}(\lambda t)$, we have by the tower property of conditional expectations and the exponential series

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{\mathrm{i}uZ_t}\right] &= \mathbb{E}\left[\mathbb{E}\left[\mathrm{e}^{\mathrm{i}uZ_t} \,\middle|\, N_t\right]\right] = \mathbb{E}\left[\mathbb{E}\left[\mathrm{e}^{\mathrm{i}u\sum_{j=1}^{k} X_j}\right]\Big|_{k=N_t}\right] \\ &= \mathbb{E}\left[\varphi_X(u)^{N_t}\right] = \sum_{k=0}^{\infty} \frac{(\varphi_X(u)\lambda t)^k}{k!}\mathrm{e}^{-\lambda t} = \mathrm{e}^{\varphi_X(u)\lambda t - \lambda t} \\ &= \mathrm{e}^{\lambda t(\varphi_X(u)-1)}. \end{aligned} \tag{2}$$

(b) Denote by $(S_k)_{k\in\mathbb{N}}$ the sequence of successive jump times of N. Then we have for all $0 \le r < t$

$$Z_t - Z_r = \Big(\sum_{k=1}^{\infty} X_k \mathbf{1}(S_k \le t)\Big) - \Big(\sum_{k=1}^{\infty} X_k \mathbf{1}(S_k \le r)\Big) = \sum_{k=1}^{\infty} X_k \mathbf{1}(r < S_k \le t)\,. \tag{3}$$

Note that the above sums are for all ω finite, and so the rearrangement is justified. Next, fix $t > 0$ and let $0 = t_0 < t_1 < \cdots < t_n = t$ and $w_1, \ldots, w_n \in \mathbb{R}$. Define the function $f : \mathbb{R} \times (0, t] \to \mathbb{R}$ by

$$f(x, s) := \sum_{j=1}^{n} w_j x \mathbf{1}(t_{j-1} < s \le t_j)\,. \tag{4}$$

Note the following simple identity:

$$\mathrm{e}^{\mathrm{i}f(x,s)} - 1 = \sum_{j=1}^{n} \left(\mathrm{e}^{\mathrm{i}w_j x} - 1\right) \mathbf{1}(t_{j-1} < s \le t_j)\,. \tag{5}$$

Moreover, we have

$$\sum_{j=1}^{n} w_j(Z_{t_j} - Z_{t_{j-1}}) = \sum_{k=1}^{\infty} f(X_k, S_k). \tag{6}$$

To simplify the notation, we may assume (after possibly enlarging the original probability space) that there exists a sequence $(U_k)_{k\in\mathbb{N}}$ of i.i.d. random variables which are uniformly distributed on $(0, t)$ and independent of $(X_k)_{k\in\mathbb{N}}$. Then for all $m \in \mathbb{N}$, by the order statistics property of the Poisson process and by invariance of $\sum_{k=1}^{m} f(X_k, S_k)$ under permutations of the indices, the conditional distribution of $\sum_{k=1}^{m} f(X_k, S_k)$ given $N_t = m$ is equal to the distribution of $\sum_{k=1}^{m} f(X_k, U_k)$. Using this, the tower property of conditional expectations and (5), we get

$$\begin{aligned}
\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\left(\sum_{j=1}^{n} w_j (Z_{t_j} - Z_{t_{j-1}})\right)}\right] &= \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\sum_{k=1}^{\infty} f(X_k,S_k)}\right] = \mathbb{E}\left[\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\sum_{k=1}^{\infty} f(X_k,S_k)} \,\middle|\, N_t\right]\right] \\
&= \sum_{m=0}^{\infty} \frac{(\lambda t)^m}{m!} \mathrm{e}^{-\lambda t} \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\sum_{k=1}^{\infty} f(X_k,S_k)} \,\middle|\, N_t = m\right] \\
&= \sum_{m=0}^{\infty} \frac{(\lambda t)^m}{m!} \mathrm{e}^{-\lambda t} \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\sum_{k=1}^{m} f(X_k,S_k)} \,\middle|\, N_t = m\right] \\
&= \sum_{m=0}^{\infty} \frac{(\lambda t)^m}{m!} \mathrm{e}^{-\lambda t} \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\sum_{k=1}^{m} f(X_k,U_k)}\right] \\
&= \sum_{m=0}^{\infty} \frac{(\lambda t)^m}{m!} \mathrm{e}^{-\lambda t} \mathbb{E}\left[\mathrm{e}^{\mathrm{i} f(X_1,U_1)}\right]^m \\
&= \mathrm{e}^{\lambda t (\mathbb{E}[\mathrm{e}^{\mathrm{i} f(X_1,U_1)}]-1)} = \mathrm{e}^{\lambda t \mathbb{E}[\mathrm{e}^{\mathrm{i} f(X_1,U_1)}-1]} \\
&= \mathrm{e}^{\lambda t \mathbb{E}[\sum_{j=1}^{n} (\mathrm{e}^{\mathrm{i} w_j X_1} - 1)\mathbf{1}(t_{j-1} < U_1 \leq t_j)]} \\
&= \mathrm{e}^{\lambda t \sum_{j=1}^{n} \mathbb{E}[\mathrm{e}^{\mathrm{i} w_j X_1} - 1]\mathbb{E}[\mathbf{1}(t_{j-1} < U_1 \leq t_j)]} \\
&= \prod_{j=1}^{n} \mathrm{e}^{\lambda (t_j - t_{j-1}) \mathbb{E}[\mathrm{e}^{\mathrm{i} w_j X_1} - 1]} = \prod_{j=1}^{n} \mathrm{e}^{\lambda (t_j - t_{j-1})(\varphi_X(w_j) - 1)}. \qquad (7)
\end{aligned}$$

Comparing this with (2) shows that Z has stationary and independent increments.

(c) It follows immediately from the definition that Z in this case is a counting process and increases by jumps of size 1. Moreover, by part (b), Z has stationary and independent increments. Therefore it remains to check that for all $t > 0$, Z_t is Poisson-distributed with parameter $p\lambda t$. Indeed, with the notation from part (a) we have

$$\varphi_X(u) = p\mathrm{e}^{iu} + (1-p) = 1 + p(\mathrm{e}^{iu} - 1). \qquad (8)$$

Hence, by part (a) we have

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}uZ_t}\right] = \mathrm{e}^{\lambda t(\varphi_X(u)-1)} = \mathrm{e}^{p\lambda t(\mathrm{e}^{iu}-1)}. \qquad (9)$$

But this is exactly the characteristic function of a Poisson-distributed random variable with parameter $p\lambda t$.

Solution 3.3

As showed during the lecture, $\mathbb{P}[T_1 > t] = \mathbb{P}[N_t = 0] = \mathrm{e}^{-\lambda t}$ for $t > 0$. This implies that $T_1 = S_1$ is $\operatorname{Exp}(\lambda)$-distributed and therefore almost surely finite.

Let $k \in \mathbb{N}$ and $0 \leqslant s_1 \leqslant t_1 \leqslant s_2 \leqslant t_2 \leqslant \ldots \leqslant s_k \leqslant t_k < \infty$. We get

$$\begin{aligned}
&\mathbb{P}[s_1 < S_1 \leqslant t_1, s_2 < S_2 \leqslant t_2, \ldots, s_k < S_k \leqslant t_k] \\
&\quad = \mathbb{P}\left[N_{s_1} = 0, N_{t_1} - N_{s_1} = 1, N_{s_2} - N_{t_1} = 0, N_{t_2} - N_{s_2} = 1, \ldots, N_{s_k} - N_{t_{k-1}} = 0, N_{t_k} - N_{s_k} \geqslant 1\right] \\
&\quad = \mathrm{e}^{-\lambda s_1} \lambda (t_1 - s_1) \mathrm{e}^{-\lambda(t_1 - s_1)} \mathrm{e}^{-\lambda(s_2 - t_1)} \lambda (t_2 - s_2) \mathrm{e}^{-\lambda(t_2 - s_2)} \ldots \mathrm{e}^{-\lambda(s_k - t_{k-1})} \left(1 - \mathrm{e}^{-\lambda(t_k - s_k)}\right) \\
&\quad = \lambda^{k-1} \left(\mathrm{e}^{-\lambda s_k} - \mathrm{e}^{-\lambda t_k}\right) \prod_{i=1}^{k-1} (t_i - s_i) \\
&\quad = \int_{s_k}^{t_k} \int_{s_{k-1}}^{t_{k-1}} \cdots \int_{s_1}^{t_1} \lambda^k \mathrm{e}^{-\lambda y_k} dy_1 dy_2 \ldots dy_k.
\end{aligned}$$

We prove by induction that the S_i's are $\mathbb{P}$-a.s. finite.

Assume that $S_1, S_2, \ldots, S_{k-1}$ are $\mathbb{P}$-a.s. finite. In a similar way as above, we have

$$\mathbb{P}\left[s_1 < S_1 \leqslant t_1, s_2 < S_2 \leqslant t_2, \ldots, s_k < S_k\right] = \lambda^{k-1} \mathrm{e}^{-\lambda s_k} \prod_{i=1}^{k-1} (t_i - s_i),$$

which converges to 0 as s_k goes to ∞. So we have

$$\mathbb{P}\left[s_1 < S_1 \leqslant t_1, s_2 < S_2 \leqslant t_2, \ldots, s_{k-1} < S_{k-1} \leqslant t_{k-1}, S_k = \infty\right] = 0.$$

Set $s_1 = 0$, $t_i = s_{i+1}$ for $i \in \{1, \ldots, k-2\}$, let t_{k-1} go to ∞, and summing over $(s_2, s_3, \ldots, s_{k-1}) \in \mathbb{Q}^{k-2}$ we get

$$\mathbb{P}\left[0 < S_1 < S_2 < \ldots < S_{k-1} < \infty, S_k = \infty\right] = 0.$$

Since $S_1, S_2, \ldots, S_{k-1}$ are $\mathbb{P}$-a.s. finite by induction hypothesis, we conclude that S_k is $\mathbb{P}$-a.s. finite. Therefore all the S_i's are $\mathbb{P}$-a.s. finite.

The sets $(s_1, t_1] \times (s_2, t_2] \times \ldots \times (s_k, t_k]$ such that $0 \leqslant s_1 \leqslant t_1 \leqslant s_2 \leqslant t_2 \leqslant \ldots \leqslant s_k \leqslant t_k < \infty$ generate the Borel σ algebra on $\{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid x_1 < x_2 < \ldots < x_k\}$. Therefore the density of the distribution of $(S_1, S_2, \ldots, S_k)$ is given by

$$f_{(S_1, S_2, \ldots, S_k)}(s_1, s_2, \ldots, s_k) = \lambda^k \mathrm{e}^{-\lambda y_k} \mathbf{1}\left(s_1 < s_2 < \ldots < s_k\right).$$

The proof that the T_i's are i.i.d. $\mathrm{Exp}(\lambda)$-distributed was done in the lecture.