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Applied Stochastic Processes

Solution Sheet 5

Solution 5.1

(a) Let us define the random variables $\widetilde{S}_n = S_n - S_0$ for $n \geqslant 1$ and the process $\widetilde{N}$ by $\widetilde{N}_t := \sum_{k=1}^{\infty} \mathbf{1}\left(\widetilde{S}_k \leqslant t\right)$. We have $\widetilde{N}_t \geqslant N_t$ for all $t \geqslant 0$, $\mathbb{P}$-almost surely. Therefore, by Lemma 2 (ii) of the lecture notes, it holds that $\mathbb{E}\left[N_t^r\right] < \infty$ for any $t \in \mathbb{R}^+$ and any $r \in \mathbb{N}$.

(b) By construction of N, it is nondecreasing, and the monotonicity property of expectaions yields that M is nondecreasing as well.

Let $0 < s < t \in \mathbb{R}^+$, we have $N_{s+\delta} \leqslant N_t$ for all $0 \leqslant \delta < t - s$. By the previous question, N_t is integrable. N is right-continuous by construction and we have $\lim_{\delta \downarrow 0} N_{s+\delta} = N_s$ $\mathbb{P}$-a.s. Using the dominated convergence theorem, we conclude that $\lim_{\delta \downarrow 0} M(s+\delta) = M(s)$. This proves that M is right-continous.

By the monotone convergence theorem, we have

$$\begin{aligned}
M(t) &= \mathbb{E}\left[N_t\right] \\
&= \mathbb{E}\left[\sum_{k=1}^{\infty} \mathbf{1}\left(S_k \leqslant t\right)\right] \\
&= \mathbb{E}\left[\lim_{n\to\infty} \sum_{k=1}^{n} \mathbf{1}\left(S_k \leqslant t\right)\right] \\
&= \lim_{n\to\infty} \mathbb{E}\left[\sum_{k=1}^{n} \mathbf{1}\left(S_k \leqslant t\right)\right] \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{E}\left[\mathbf{1}\left(S_k \leqslant t\right)\right] \\
&= \sum_{k=1}^{\infty} \mathbb{P}\left[S_k \leqslant t\right] \\
&= \sum_{k=1}^{\infty} G * F^{*k}(t).
\end{aligned}$$

This proves the claim.

(c) Let $s > 0$, we have similarly to the proof from the lecture,

$$\begin{aligned}
\hat{M}(s) &= \int_0^\infty \mathrm{e}^{-sx} dM(x) \\
&= \int_0^\infty \mathrm{e}^{-sx} d \sum_{k=1}^\infty G * F^{*k}(x) \\
&= \sum_{k=1}^\infty \int_0^\infty \mathrm{e}^{-sx} d \left(G * F^{*k} \right)(x) \\
&= \sum_{k=1}^\infty \mathbb{E}\left[\mathrm{e}^{-sS_k}\right] \\
&= \sum_{k=1}^\infty \mathbb{E}\left[\mathrm{e}^{-sS_0}\right] \mathbb{E}\left[\mathrm{e}^{-sT_1}\right]^k \\
&= \sum_{k=1}^\infty \hat{G}(s)\hat{F}(s)^k \\
&= \hat{G}(s)\frac{\hat{F}(s)}{1-\hat{F}(s)},
\end{aligned}$$

where we first used the definition of the Laplace transform, the independence of the T_i's, that they are identically distributed and the independence of S_0 from the T_i's. For $s = 0$ both sides of the equality are infinite.

(d) Let $t \geqslant 0$. We have from (b) the equality $M(t) = \sum_{k=1}^\infty G * F^{*k}(t)$. Rewriting,

$$\begin{aligned}
M(t) &= G * F(t) + \sum_{k=2}^\infty G * F^{*k}(t) \\
&= G * F(t) + \sum_{k=2}^\infty \int_0^t G * F^{*k-1}(t-s) dF(s) \\
&= G * F(t) + \sum_{k=1}^\infty \int_0^t G * F^{*k}(t-s) dF(s) \\
&= G * F(t) + \int_0^t M(t-s) dF(s).
\end{aligned}$$

For $t < 0$, it holds that $M(t) = 0$.

Solution 5.2

Observe that the probability distribution of the interarrival times T_n is Gamma$(\lambda, 2)$, which is the convolution of two Exp(λ) distributions. The interarrival times can thus be written as $T_n = V_{2n-1} + V_{2n}$, $n \in \mathbb{N}$ where V_n are independent and exponentially distributed random variables with rate λ.

Let $(\tilde{N}_t)_{t\geq 0}$ be the renewal process with interarrival times V_n, $n \in \mathbb{N}$. Hence, for all $n \in \mathbb{N}_0$,

$$\begin{aligned}
P[N_t = n] &= P\left[\tilde{N}_t = 2n\right] + P\left[\tilde{N}_t = 2n+1\right] \\
&= \mathrm{e}^{-\lambda t}\left(\frac{(\lambda t)^{2n}}{(2n)!} + \frac{(\lambda t)^{2n+1}}{(2n+1)!} \right)
\end{aligned}$$

and

$$\begin{aligned}
M(t) = \mathbb{E}[N_t] &= \sum_{n=0}^{\infty} nP\left[N_t = n\right] \\
&= \mathrm{e}^{-\lambda t}\frac{1}{2}\sum_{n=0}^{\infty}\left(2n\frac{(\lambda t)^{2n}}{(2n)!} + (2n+1)\frac{(\lambda t)^{2n+1}}{(2n+1)!} - \frac{(\lambda t)^{2n+1}}{(2n+1)!}\right) \\
&= \frac{\mathrm{e}^{-\lambda t}}{2}\sum_{n=0}^{\infty} n\frac{(\lambda t)^n}{n!} - \frac{\mathrm{e}^{-\lambda t}}{2}\sum_{n=0}^{\infty}\frac{(\lambda t)^{2n+1}}{(2n+1)!} \\
&= \frac{\lambda t}{2} - \frac{\mathrm{e}^{-\lambda t}}{2}\left(\frac{\mathrm{e}^{\lambda t} - \mathrm{e}^{-\lambda t}}{2}\right) \\
&= \frac{\lambda t}{2} - \frac{1 - \mathrm{e}^{-2\lambda t}}{4}.
\end{aligned}$$

From this we obtain $\lim_{t\to\infty} \frac{M(t)}{t} = \frac{\lambda}{2}$.

Solution 5.3

(a) Let $S_k = T_1 + \ldots + T_k$. For $t \geq 0$,

$$\begin{aligned}
g(t) &= P[Y_t = 1] \\
&= P[Y_t = 1, T_1 > t] + P[Y_t = 1, T_1 \leq t] \\
&= P[U_1 > t] + E\left[\textstyle\sum_{k\geq 1} \mathbb{1}_{\{S_k \leq t < S_k + U_{k+1}\}}\right] \\
&= P[U_1 > t] + E\left[\textstyle\sum_{k\geq 1} \mathbb{1}_{\{T_1 + S_k - S_1 \leq t < T_1 + S_k - S_1 + U_{k+1}\}}\right],
\end{aligned}$$

where T_1 is independent of $S_k - S_1$ and of U_{k+1}, and $S_k - S_1 \overset{\mathcal{D}}{=} S_{k-1}$ for $k \geq 1$. By conditioning on T_1, we obtain

$$\begin{aligned}
&E\left[\textstyle\sum_{k\geq 1} \mathbb{1}_{\{T_1 + S_k - S_1 \leq t < T_1 + S_k - S_1 + U_{k+1}\}}\middle| T_1\right] \\
=&E\left[\textstyle\sum_{k\geq 1} \mathbb{1}_{\{S_k - S_1 \leq t - s < S_k - S_1 + U_{k+1}\}}\right]\Bigg|_{s=T_1} = g(t - T_1) \quad \text{a.e. on } \{T_1 \leq t\}.
\end{aligned}$$

Hence, using the tower property,

$$g(t) = P[U_1 > t] + E[g(t - T_1)\mathbb{1}_{\{T_1 \leq t\}}] = P[U_1 > t] + \int_0^t g(t-s)\,\mathrm{d}F(s), \quad t \geq 0.$$

(b) Note that T_1 is the sum of two independent random variables U_1 and V_1, which have densities $u(t) = \lambda \mathrm{e}^{-\lambda t}\mathbb{1}_{\{t\geq 0\}}$ and $v(t) = \mu \mathrm{e}^{-\mu t}\mathbb{1}_{\{t\geq 0\}}$, respectively. Hence T_1 also has a density f, which is given by $f = u * v$. In particular, f is supported on $[0, \infty)$ and for $t \geq 0$ we have

$$\begin{aligned}
f(t) = (u * v)(t) &= \int_0^t u(s)v(t-s)\,\mathrm{d}s = \lambda\mu \int_0^t \mathrm{e}^{-\lambda s - \mu(t-s)}\,\mathrm{d}s \\
&= \lambda\mu \mathrm{e}^{-\mu t}\int_0^t \mathrm{e}^{(\mu-\lambda)s}\,\mathrm{d}s = \frac{\lambda\mu}{\mu - \lambda}\mathrm{e}^{-\mu t}(\mathrm{e}^{(\mu-\lambda)t} - 1) \\
&= \frac{\lambda\mu}{\mu - \lambda}(\mathrm{e}^{-\lambda t} - \mathrm{e}^{-\mu t}).
\end{aligned}$$

Next, by a) we know that g satisfies the renewal equation

$$g(t) = \mathbb{P}[U_1 > t] + \int_0^t g(t-s) f(s) \, \mathrm{d}s, \quad t \geq 0.$$

Observing that $\mathbb{P}[U_1 > t] = \mathrm{e}^{-\lambda t} \mathbb{1}_{\{t \geq 0\}} = h(t)$ yields the claim.