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D-MATH
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Applied Stochastic Processes

Solution Sheet 7

Solution 7.1

The stochastic processes described in a) and b) are Markov chains, while the one in c) is not. Let Y_n denote the number which shows up in the n-th roll.

(a) We have $X_n = (X_{n-1}+1)\,\mathbb{1}_{\{Y_n<6\}}$. Thus, $(X_n)_{n\in\mathbb{N}}$ is a Markov chain with state space $\mathbb{N}_0$. For $i,j \in \{0,1,2,\dots\}$:

$$r_{i,j} = \begin{cases} \frac{1}{6} & \text{if } j = 0, \\ \frac{5}{6} & \text{if } j = i+1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Then $X_n = \max\{X_{n-1}, Y_n\}$. Hence, $(X_n)_{n\in\mathbb{N}}$ is a Markov chain with state space $\{1,\dots,6\}$. We obtain the following transition probabilities for $1 \le i,j \le 6$:

$$r_{i,j} = \begin{cases} 0 & \text{if } j < i, \\ \frac{i}{6} & \text{if } j = i, \\ \frac{1}{6} & \text{if } j > i. \end{cases}$$

Furthermore, noting that $r_{i,j}(n) = P\big[\max\{Y_1,Y_2,\dots,Y_n\} = j \mid X_0 = i\big]$ for $j > i$, we have

$$r_{i,j}(n) = \begin{cases} 0 & \text{if } j < i, \\ \left(\frac{i}{6}\right)^n & \text{if } j = i, \\ \left(\frac{j}{6}\right)^n - \left(\frac{j-1}{6}\right)^n & \text{if } j > i. \end{cases}$$

(c) The transition probabilities at time n depend not only on X_n, but also on X_{n-1}. For example,

$$\begin{aligned} \mathbb{P}[X_4 = 6 | X_3 = 6] = \mathbb{P}[Y_3 = 6 | X_3 = 6] + \mathbb{P}[Y_3 < 6, Y_4 = 6 | X_3 = 6] = \frac{6}{11} + \frac{5}{11}\cdot\frac{1}{6} \\ < 1 = \mathbb{P}[X_4 = 6 | X_3 = 6, X_2 = 1]. \end{aligned}$$

Therefore, this is not a Markov chain.

Solution 7.2

(a) The transition matrix is given by

$$R = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}.$$

(b) Fix $n \in \mathbb{N}$. If $X_n = 0$, then $X_{n+1} = 1$ and if $X_n = N$ then $X_{n+1} = N - 1$. If $X_n = i$, where $i \in \{1, \dots, N-1\}$, then we have $X_{n+1} \in \{i-1, i, i+1\}$ with

$$\begin{aligned} r_{i,i-1} &= \frac{i^2}{N^2}, \\ r_{i,i} &= \frac{i(N-i) + (N-i)i}{N^2} = \frac{2i(N-i)}{N^2}, \\ r_{i,i+1} &= \frac{(N-i)^2}{N^2}. \end{aligned}$$

Thus the transition matrix is

$$R = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ \frac{1}{N^2} & \frac{2(N-1)}{N^2} & \frac{(N-1)^2}{N^2} & 0 & & & \vdots \\ 0 & \frac{4}{N^2} & \frac{4(N-2)}{N^2} & \frac{(N-2)^2}{N^2} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \frac{(N-2)^2}{N^2} & \frac{4(N-2)}{N^2} & \frac{4}{N^2} & 0 \\ \vdots & & & 0 & \frac{(N-1)^2}{N^2} & \frac{2(N-1)}{N^2} & \frac{1}{N^2} \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

(c) We have $\mathbb{P}[Y_n = 1] = p$ and $\mathbb{P}[Y_n = 0] = 1 - p =: q$, $n \in \mathbb{N}$. If we identify X_n with the corresponding binary number $\sum_{i=1}^{k} Y_{n+i} 2^{k-i}$, the state space of $(X_n)_{n \in \mathbb{N}}$ is given by $\{0, 1, 2, \dots, 2^k - 1\}$. Using this representation of X_n we have

$$\begin{aligned} X_{n+1} &= \sum_{i=1}^{k} Y_{n+1+i} 2^{k-i} = Y_{n+k+1} + \sum_{i=2}^{k} Y_{n+i} 2^{k-i+1} \\ &= Y_{n+k+1} + \sum_{i=1}^{k} Y_{n+i} 2^{k-i+1} - Y_{n+1} 2^k = Y_{n+k+1} + 2X_n - Y_{n+1} 2^k \\ &= Y_{n+k+1} + 2X_n \mod 2^k. \end{aligned}$$

Hence, we have

$$X_{n+1} = \begin{cases} 2X_n + 1 \mod 2^k & \text{with probability } p, \\ 2X_n \mod 2^k & \text{with probability } q. \end{cases}$$

The corresponding transition matrix is thus

$$R = \left.\begin{pmatrix} q & p & 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & q & p & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & q & p & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & q & p & 0 & 0 \\ & & & & & & & 0 & 0 & q & p \\ q & p & 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & q & p & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & q & p & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & q & p & 0 & 0 \\ & & & & & & & 0 & 0 & q & p \end{pmatrix}\right\} 2^k \text{ rows and columns.}$$

Solution 7.3

(a) Let $f \in L^\infty(E)$. There exists $K > 0$ such that $||f||_\infty = K$. Let $n \in \mathbb{N}$. By definition of $R(n)$ and the conditional expectation, we have

$$(R(n)f)(x) := \begin{cases} \mathbb{E}\left[f(X_n) \mid X_{n-1} = x\right] \leqslant K, & \text{for } x \in E \text{ if } \mathbb{P}[X_{n-1} = x] > 0, \\ f(x) \leqslant K & \text{for } x \in E \text{ if } \mathbb{P}[X_{n-1} = x] = 0. \end{cases}$$

The choice of f was arbitrary. Then, by definition of the norm of an operator, we have

$$||R(n)|| = \sup_{f \in L^\infty(E), ||f||=1} ||R(n)f|| \leqslant 1.$$

.

(b) $(X_n)_{n\in\mathbb{N}}$ is by definition a discrete time Markov chain if and only if for all $n \in \mathbb{N}$ and all bounded functions $f : E \to \mathbb{R}$ we have

$$\mathbb{E}[f(X_{n+1}) \,|\, X_0, \ldots, X_n] = \mathbb{E}[f(X_{n+1}) \,|\, X_n] \ \ \mathbb{P}\text{-a.s.} \tag{1}$$

Therefore, to establish both directions it suffices to show that $(R(n+1)f)_{X_n}$ is a version of the conditional expectation $\mathbb{E}[f(X_{n+1}) \,|\, X_n]$ for all $n \in \mathbb{N}$ and all bounded functions $f : E \to \mathbb{R}$. Fix $n \in \mathbb{N}$ and a bounded function $f : E \to \mathbb{R}$. Since E is at most countable and $\mathcal{E} = 2^E$, any function $E \to \mathbb{R}$ is measurable, which implies that $(R(n+1)f)_{X_n}$ is $\sigma(X_n)$-measurable. To establish the averaging property, let $A \in \sigma(X_n)$. Since E is countable, we may assume without loss of generality that $A = \{X_n = x\}$ for some $x \in E$. If $\mathbb{P}[X_n = x] > 0$, we have

$$\begin{aligned} \mathbb{E}[\mathbb{1}_A \mathbb{E}[f(X_{n+1}) \,|\, X_n]] &= \mathbb{E}[\mathbb{1}_A f(X_{n+1})] = \mathbb{E}[\mathbb{1}_{\{X_n = x\}} f(X_{n+1})] \\ &= \sum_{y\in E} \mathbb{P}[X_n = x, X_{n+1} = y] f(y) \\ &= \sum_{y\in E} \mathbb{P}[X_n = x]\mathbb{P}[X_{n+1} = y \,|\, X_n = x] f(y) \\ &= \mathbb{P}[X_n = x] \sum_{y\in E} R_{x,y}(n+1) f(y) \\ &= \mathbb{P}[X_n = x](R(n+1)f)_x \\ &= \mathbb{E}[\mathbb{1}_{\{X_n = x\}}(R(n+1)f)_x] \\ &= \mathbb{E}[\mathbb{1}_{\{X_n = x\}}(R(n+1)f)_{X_n}] \\ &= \mathbb{E}[\mathbb{1}_A (R(n+1)f)_{X_n}], \end{aligned} \tag{2}$$

and if $\mathbb{P}[X_n = x] = 0$, we have

$$\begin{aligned}\mathbb{E}[\mathbb{1}_A \mathbb{E}[f(X_{n+1})|X_n]] &= \mathbb{E}[\mathbb{1}_A f(X_{n+1})] = \mathbb{E}[\mathbb{1}_{\{X_n=x\}} f(X_{n+1})] = 0 \\ &= \mathbb{E}[\mathbb{1}_{\{X_n=x\}}(R(n+1)f)_x] = \mathbb{E}[\mathbb{1}_{\{X_n=x\}}(R(n+1)f)_{X_n}] \\ &= \mathbb{E}[\mathbb{1}_A (R(n+1)f)_{X_n}]. \end{aligned} \tag{3}$$

(c) First, suppose that $(X_n)_{n\in\mathbb{N}}$ is a discrete time Markov chain. We prove the stated equation by induction on n. The basis $n = 0$ is trivial. For the induction hypothesis assume that we have shown the claim for $n \in \mathbb{N}$. Let $x_0, \dots, x_{n+1} \in E$. Using part (b) with $f = \mathbb{1}_{x_{n+1}}$, the averaging property of conditional expectations and the induction hypothesis, we get

$$\begin{aligned}\mathbb{P}[X_0 = x_0, \dots, X_{n+1} = x_{n+1}] &= \mathbb{E}[\mathbb{1}_{\{X_0=x_0\}} \times \cdots \times \mathbb{1}_{\{X_n=x_n\}} f(X_{x+1})] \\ &= \mathbb{E}[\mathbb{E}[\mathbb{1}_{\{X_0=x_0\}} \times \cdots \times \mathbb{1}_{\{X_n=x_n\}} f(X_{x+1}) \,|\, X_0, \dots, X_n]] \\ &= \mathbb{E}[\mathbb{1}_{\{X_0=x_0\}} \times \cdots \times \mathbb{1}_{\{X_n=x_n\}} \mathbb{E}[f(X_{x+1}) \,|\, X_0, \dots, X_n]] \\ &= \mathbb{E}[\mathbb{1}_{\{X_0=x_0\}} \times \cdots \times \mathbb{1}_{\{X_n=x_n\}} (R(n+1)f)_{X_n}] \\ &= \mathbb{E}[\mathbb{1}_{\{X_0=x_0\}} \times \cdots \times \mathbb{1}_{\{X_n=x_n\}} R_{X_n,x_{n+1}}(n+1)] \\ &= \mathbb{E}[\mathbb{1}_{\{X_0=x_0\}} \times \cdots \times \mathbb{1}_{\{X_n=x_n\}} R_{x_n,x_{n+1}}(n+1)] \\ &= \mathbb{E}[\mathbb{1}_{\{X_0=x_0\}} \times \cdots \times \mathbb{1}_{\{X_n=x_n\}}] R_{x_n,x_{n+1}}(n+1) \\ &= \mathbb{P}[X_0 = x_0, \dots, X_n = x_n] R_{x_n,x_{n+1}}(n+1) \\ &= \mu_{x_0} R_{x_0,x_1}(1) \times \cdots \times R_{x_{n-1},x_n}(n) R_{x_n,x_{n+1}}(n+1). \end{aligned} \tag{4}$$

Conversely, suppose that the stated condition holds. By part (b) and *Dynkin's lemma* using that E is countable, it suffices to show that for all $x_0, \dots, x_n \in E$ and all bounded functions $f : E \to \mathbb{R}$ we have

$$\mathbb{E}[\mathbb{1}_{\{X_0=x_0\}} \times \cdots \times \mathbb{1}_{\{X_n=x_n\}} f(X_{n+1})] = \mathbb{E}[\mathbb{1}_{\{X_0=x_0\}} \times \cdots \times \mathbb{1}_{\{X_n=x_n\}} (R(n+1)f)_{X_n}] \tag{5}$$

Again, by Dynkin's lemma using that E is countable, we may assume without loss of generality that $f = \mathbb{1}_{x_{n+1}}$ for some $x_{n+1} \in E$. Fix $x_0, \dots, x_{n+1} \in E$. Then we have

$$\begin{aligned}\mathbb{E}[\mathbb{1}_{\{X_0=x_0\}} \times \cdots \times \mathbb{1}_{\{X_n=x_n\}} f(X_{n+1})] &= \mathbb{P}[X_0 = x_0, \dots, X_{n+1} = x_{n+1}] \\ &= \mu_{x_0} R_{x_0,x_1}(1) \times \cdots \times R_{x_{n-1},x_n}(n) R_{x_n,x_{n+1}}(n+1) \\ &= \mathbb{P}[X_0 = x_0, \dots, X_n = x_n] R_{x_n,x_{n+1}}(n+1) \\ &= \mathbb{E}[\mathbb{1}_{\{X_0=x_0\}} \times \cdots \times \mathbb{1}_{\{X_n=x_n\}} R_{x_n,x_{n+1}}(n+1)] \\ &= \mathbb{E}[\mathbb{1}_{\{X_0=x_0\}} \times \cdots \times \mathbb{1}_{\{X_n=x_n\}} R_{X_n,x_{n+1}}(n+1)] \\ &= \mathbb{E}[\mathbb{1}_{\{X_0=x_0\}} \times \cdots \times \mathbb{1}_{\{X_n=x_n\}} (R(n+1)f)_{X_n}]. \end{aligned} \tag{6}$$

(d) Fix $n \in \mathbb{N}$. Using part (c), we get

$$\begin{aligned}\mathbb{E}[f(X_n)] &= \sum_{x_0\in E} \cdots \sum_{x_n\in X_n} \mathbb{P}[X_0 = x_0, \dots, X_n = x_n] f(x_n) \\ &= \sum_{x_0\in E} \cdots \sum_{x_n\in X_n} \mu_{x_0} R_{x_0,x_1}(1) \times \cdots \times R_{x_{n-1},x_n}(n) f(x_n) \\ &= \mu R(1) R(2) \cdots R(n) f. \end{aligned} \tag{7}$$

(e) First, suppose that there exists a transition matrix R such that for all $n \in \mathbb{N}$ and all $y \in E$ we have

$$R_{x,y} = R_{x,y}(n+1) \quad \text{if} \quad \mathbb{P}[X_n = x] > 0. \tag{8}$$

But this implies that for all $y \in Y$ we have

$$(R\mathbb{1}_y)_{X_n} = R_{X_n,y} = R_{X_n,y}(n+1) = (R(n+1)\mathbb{1}_y)_{X_n} \ \ \mathbb{P}\text{-a.s.} \tag{9}$$

By Dynkin's lemma it follows that for all bounded functions $f : E \to \mathbb{R}$ we have

$$(Rf)_{X_n} = (R(n+1)f)_{X_n} \;\; \mathbb{P}\text{-a.s.} \tag{10}$$

Since $(R(n+1)f)_{X_n} = \mathbb{E}[f(X_{n+1}) \,|\, X_0, \dots, X_n]$ $\mathbb{P}$-a.s., we may conclude that $(X_n)_{n\in\mathbb{N}}$ is a homogeneous Markov chain.

Conversely, suppose that $(X_n)_{n\in\mathbb{N}}$ is a homogeneous Markov chain. Then there exists a transition matrix R such that for all $n \in \mathbb{N}$ and all bounded functions $f : E \to \mathbb{R}$ we have

$$\mathbb{E}[f(X_{n+1}) \,|\, X_0, \dots, X_n] = (Rf)_{X_n} \;\; \mathbb{P}\text{-a.s.} \tag{11}$$

Let $n \in \mathbb{N}$, $x, y \in E$ with $\mathbb{P}[X_0 = x] > 0$. Then by the averaging property of conditional expectations we get with $f = \mathbb{1}_y$

$$\begin{aligned} R_{x,y} = (R\mathbb{1}_y)_x &= \frac{\mathbb{E}[\mathbb{1}_{\{X_n=x\}}(R\mathbb{1}_y)_x]}{\mathbb{P}[X_n = x]} = \frac{\mathbb{E}[\mathbb{1}_{\{X_n=x\}}(Rf)_{X_n}]}{\mathbb{P}[X_n = x]} \\ &= \frac{\mathbb{E}[\mathbb{1}_{\{X_n=x\}}f(X_{n+1})]}{\mathbb{P}[X_n = x]} = \frac{\mathbb{P}[X_{n+1} = y, X_n = x]}{\mathbb{P}[X_n = x]} \\ &= \mathbb{P}[X_{n+1} = y \,|\, X_n = x] = R_{x,y}(n+1) \end{aligned} \tag{12}$$

This establishes the claim.

Solution 7.4

(a) First, note that after the first jump either a ruin occurs, or the risk process f_x continues as if started at time 0 from initial state (capital) $x + cS_1 - X_1 \geq 0$ (since X_i are i.i.d. and independent of arrival process). Denote by H the distribution function of S_1, so that $\mathrm{d}H(s) = \lambda e^{-\lambda s}\,\mathrm{d}s$. By the law of total probability, conditioning on the time S_1 and size X_1 of the first jump, we have

$$\begin{aligned} R(x) &= \mathbb{P}[f_x(t) \geq 0 \text{ for all } t > 0, x + cS_1 - X_1 > 0] \\ &= \int_{\{(s,y):x+cs-y\geq 0\}} \mathbb{P}[f_{x+cs-y}(t) \geq 0 \text{ for all } t > 0] \;\mathrm{d}G(y)\,\mathrm{d}H(s) \\ &= \int_0^\infty \int_0^{x+cs} R(x+cs-y)\lambda e^{-\lambda s} \;\mathrm{d}G(y)\,\mathrm{d}s \end{aligned}$$

by independence of X_1 and S_1. Substitute $u = x + cs$ to obtain

$$R(x) = \int_x^\infty \int_0^u R(u-y)\frac{\lambda}{c}e^{-\lambda(u-x)/c} \;\mathrm{d}G(y)\,\mathrm{d}u.$$

Multiply both sides by $e^{-\lambda x/c}$ and rearrange:

$$e^{-\lambda x/c}R(x) = \frac{\lambda}{c}\int_{u=x}^\infty e^{-\lambda u/c}\left(\int_{y=0}^u R(u-y)\,\mathrm{d}G(y)\right)\mathrm{d}u.$$

The right-hand side is the integral of a bounded function, thus it is continuous on $(0, \infty)$, and hence so is the left-hand side, in particular R. Then, in turn, the integrand on the RHS is continuous, hence the integral is differentiable. Differentiating both sides w.r.t. x yields

$$e^{-\lambda x/c}R'(x) - \frac{\lambda}{c}e^{-\lambda x/c}R(x) = -\frac{\lambda}{c}e^{-\lambda x/c}\int_0^x R(x-y)\;\mathrm{d}G(y). \tag{13}$$

Using $R(x) = \int_0^x R'(u)\,\mathrm{d}u + R(0)$ and Fubini, we can rewrite the integral

$$\begin{aligned}
\int_0^x R(x-y)\ \mathrm{d}G(y) &= \int_0^x \int_0^{x-y} R'(u)\,\mathrm{d}u\ \mathrm{d}G(y) + R(0)\mathbb{P}[X_1 \le x] \\
&= \int_0^x \int_y^x R'(x-u)\,\mathrm{d}u\ \mathrm{d}G(y) + R(0)\mathbb{P}[X_1 \le x] \\
&= \int_0^x \left(\int_0^u \mathrm{d}G(y) \right) R'(x-u)\,\mathrm{d}u + R(0)\mathbb{P}[X_1 \le x] \\
&= \int_0^x R'(x-u)\mathbb{P}[X_1 \le u]\,\mathrm{d}u + R(0)\mathbb{P}[X_1 \le x]. \qquad (14)
\end{aligned}$$

Rearranging (13) and plugging in (14) yields

$$\begin{aligned}
R'(x) &= \frac{\lambda}{c} R(x) - \frac{\lambda}{c} \int_0^x R(x-y)\ \mathrm{d}G(y) \\
&= \frac{\lambda}{c} \left(\int_0^x R'(u)\,\mathrm{d}u + R(0) - \int_0^x R'(x-u)\mathbb{P}[X_1 \le u]\,\mathrm{d}u - R(0)\mathbb{P}[X_1 \le x] \right) \\
&= \frac{\lambda}{c}\mathbb{P}[X_1 > x]R(0) + \int_0^x R'(x-u)\frac{\lambda}{c}\mathbb{P}[X_1 > u]\,\mathrm{d}u,
\end{aligned}$$

which is the renewal equation we wanted to obtain.

(b) (i) We compute,

$$\begin{aligned}
R(x) &= \mathbb{P}\left[x + ct \ge \sum_{i=1}^{N_t} X_i,\ \forall t \right] \\
&= \mathbb{P}\left[x + cS_n \ge \sum_{i=1}^{n} X_i,\ \forall n \right] \\
&= \mathbb{P}\left[x \ge \sum_{i=1}^{n} X_i - cS_n,\ \forall n \right] \\
&= \mathbb{P}\left[x \ge \sup \left\{ \sum_{i=1}^{n} X_i - cS_n, n \in \mathbb{N} \right\} \right].
\end{aligned}$$

If we show that $\sup \left\{ \sum_{i=1}^{n} X_i - cS_n, n \in \mathbb{N} \right\} < \infty$ a.s., then it follows that $R(\infty) = 1$. Now

$$\mathbb{E}[X_i - c(S_i - S_{i-1})] = \mathbb{E}[X_1] - c/\lambda < 0,$$

so, by the strong law of large numbers,

$$\sum_{i=1}^{n} (X_i - c(S_i - S_{i-1})) = \sum_{i=1}^{n} X_i - cS_n \to -\infty \quad \text{a.s.,} \quad n \to \infty,$$

thus a finite maximum exists a.s.

(ii) We define ϕ and θ as the Laplace transforms of the r.v. X_1 and the function R' respectively:

$$\phi(u) = \mathbb{E}[\mathrm{e}^{-uX_1}], \quad \theta(u) = \int_0^\infty e^{-ux} R'(x)\,\mathrm{d}x.$$

Using the formula $\int_0^\infty \mathrm{e}^{-ux}(1-G(x))\,\mathrm{d}x = (1-\hat{G}(u))/u$ we obtain the Laplace transform version of the renewal equation computed in (a),

$$\theta(u) = \frac{\lambda}{c}R(0)(1-\phi(u))/u + \frac{\lambda}{c}(1-\phi(u))\theta(u)/u.$$

Solving for θ yields

$$\theta(u) = \frac{\lambda R(0)(1-\phi(u))/u}{c-\lambda(1-\phi(u))/u}. \tag{15}$$

Notice that $\lim_{u\downarrow 0}(1-\phi(u))/u = \mathbb{E}[X_1]$ (by MCT, since $(1-e^{-ux})/u \uparrow x$ as $u \downarrow 0\ \forall x > 0$). Hence,

$$\lim_{u\downarrow 0}\theta(u) = \int_0^\infty R'(x)\,\mathrm{d}x = \frac{\lambda\mathbb{E}[X_1]R(0)}{c-\lambda\mathbb{E}[X_1]}.$$

But since also $\int_0^\infty R'(x)\,\mathrm{d}x = R(\infty)-R(0)$, we can now solve

$$R(0) = 1-\frac{\lambda}{c}\mathbb{E}[X_1]. \tag{16}$$

(iii) Notice that (15) can be written as a sum of a geometric sequence,

$$\theta(u) = R(0)\sum_{n=1}^\infty\left(\frac{\lambda}{c}(1-\phi(u))/u\right)^n,$$

hence, using the properties of Laplace transform,

$$R'(t) = R(0)\sum_{n=1}^\infty (F')^{*n}(t).$$

Thus

$$\begin{aligned}
R(t) &= R(0) + \int_0^t R'(u)\,\mathrm{d}u \\
&= R(0)\left(1+\int_0^t\sum_{n=1}^\infty (F')^{*n}(s)\,\mathrm{d}s\right) \\
&= R(0)(1+M(t))
\end{aligned}$$

as required. We used in the last inequality, that for a distribution with density, the density of the n-th convolution is the n-th convolution of the distribution's density.

(c) Define the two functions

$$F_\alpha(t) = \int_0^t \mathrm{e}^{\alpha x}dF(x)\mathbb{1}\,(t \geqslant 0)\,,\quad h_\alpha(t) = R(0)\frac{F(t)-F(\infty)}{1-F(\infty)}\mathrm{e}^{\alpha t}\mathbb{1}\,(t\geqslant 0)$$

F_α is non-arithmetic because F is. $-h_\alpha$ is non-increasing, non-negative and we have

$$\begin{aligned}
\int_0^\infty -h_\alpha(t)dt &= \frac{R(0)\lambda}{c\,(1-F(\infty))}\int_0^\infty\int_t^\infty \mathbb{P}\left[X_1 > u\right]dudt \\
&= \frac{R(0)\lambda}{c\,(1-F(\infty))}\int_0^\infty\int_0^u \mathbb{P}\left[X_1 > u\right]dtdu \\
&= \frac{R(0)\lambda}{c\,(1-F(\infty))}\int_0^\infty u\mathbb{P}\left[X_1 > u\right]du \\
&= \frac{R(0)\lambda}{2c\,(1-F(\infty))}\mathbb{E}\left[X_1^2\right] < \infty.
\end{aligned}$$

By the criterion in the lecture h_α is DRI.
Smith's theorem for renewal equations with defect yields

$$\lim_{t\to\infty}(1-R(t))\mathrm{e}^{\alpha t}=\frac{R(0)}{\int_0^\infty x\mathrm{e}^{\alpha x}\frac{\lambda}{c}\mathbb{P}\left[X_1>x\right]dx}\int_0^\infty\frac{F(\infty)-F(t)}{1-F(\infty)}\mathrm{e}^{\alpha t}dt.$$

The right-hand side can be simplified to

$$\begin{aligned}\int_0^\infty F(\infty)-F(t)\mathrm{e}^{\alpha t}dt &= \int_0^\infty\int_t^\infty dF(u)\mathrm{e}^{\alpha t}dt\\ &= \int_0^\infty\int_0^u \mathrm{e}^{\alpha t}dtdF(u)\\ &= \int_0^\infty\frac{1}{\alpha}\left(\mathrm{e}^{u\alpha}-1\right)dF(u)\\ &= \frac{1}{\alpha}(1-F(\infty)).\end{aligned}$$

Replacing in the previous equation and using the value of $R(0)$, we get

$$1-R(t)\sim_{t\to\infty}\frac{\mathrm{e}^{-\alpha t}\left(c-\lambda\mathbb{E}\left[X_1\right]\right)}{\alpha\lambda\int_0^\infty x\mathrm{e}^{\alpha x}\mathbb{P}\left[X_1>x\right]dx}.$$