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Applied Stochastic Processes

Solution Sheet 9

Solution 9.1

(a) The number of molecules in A can only increase or decrease by 1. The probability that it increases (resp., decreases) is equal to the probability that a molecule from compartment B (resp., A) is chosen. Thus the transition probabilities are

$$r_{x,y} = \begin{cases} 1 - \frac{x}{N} & \text{if } x < N \text{ and } y = x+1, \\ \frac{x}{N} & \text{if } x > 0 \text{ and } y = x-1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) We consider the detailed balance condition, $\pi(x)r_{x,x-1} = \pi(x-1)r_{x-1,x}$, and obtain

$$\pi(x) = \pi(x-1)\frac{r_{x-1,x}}{r_{x,x-1}} = \pi(x-1)\frac{N-x+1}{x} = \pi(0)\frac{N!}{(N-x)! \cdot x!} = \pi(0)\binom{N}{x}.$$

Furthermore, $\sum_{x=0}^{N} \pi(x) = 1$, so

$$\pi(0) = \left(\sum_{x=0}^{N} \binom{N}{x}\right)^{-1} = \left((1+1)^N\right)^{-1} = 2^{-N}.$$

Since this distribution satisfies the detailed balance condition, it is reversible (and hence stationary, see lecture notes).

Solution 9.2

(a) First, suppose that $(X_n)_{n\in\mathbb{N}_0}$ is reversible. By a result of the lecture, we have for $i, j \in E$

$$\mu_i r_{i,j} = \mathbb{P}_\mu[X_0 = i, X_1 = j] = \mathbb{P}_\mu[X_0 = j, X_1 = i] = \mu_j r_{j,i},$$

so μ satisfies the detailed balance condition and is thus a reversible distribution for $(X_n)_{n\in\mathbb{N}_0}$. Conversely, suppose that μ is a reversible distribution for $(X_n)_{n\in\mathbb{N}_0}$. Let $m \in \mathbb{N}$ and $i_0, \dots, i_m \in E$. Using the same result again and the detailed balance condition m times, we obtain

$$\begin{aligned}
\mathbb{P}_\mu[X_0 = i_0, X_1 = i_1, \dots, X_m = i_m] &= \mu_{i_0} r_{i_0,i_1} r_{i_1,i_2} \cdots r_{i_{m-1},i_m} \\
&= r_{i_1,i_0} \mu_{i_1} r_{i_1,i_2} \cdots r_{i_{m-1},i_m} \\
&\ \ \vdots \\
&= r_{i_1,i_0} \cdots r_{i_{m-1},i_{m-2}} \mu_{i_{m-1}} r_{i_{m-1},i_m} \\
&= r_{i_1,i_0} \cdots r_{i_{m-1},i_{m-2}} r_{i_m,i_{m-1}} \mu_{i_m} \\
&= \mu_{i_m} r_{i_m,i_{m-1}} r_{i_{m-1},i_{m-2}} \cdots r_{i_1,i_0} \\
&= \mathbb{P}_\mu[X_0 = i_m, X_1 = i_{m-1}, \dots, X_m = i_0].
\end{aligned}$$

(b) Suppose that $(X_n)_{n\in\mathbb{N}_0}$ is reversible. Then μ is a reversible distribution for $(X_n)_{n\in\mathbb{N}_0}$ by part a). Again by part a) it suffices to show that μ' is a reversible distribution for $(X'_n)_{n\in\mathbb{N}_0}$. To this end, we have to check the detailed balance condition. Let $i,j\in F$. Note that we only have to consider the case $i\neq j$. Since μ satisfies the detailed balance condition, we obtain

$$\mu'_i r'_{i,j} = \frac{\mu_i r_{i,j}}{\sum_{k\in F}\mu_k} = \frac{\mu_j r_{j,i}}{\sum_{k\in F}\mu_k} = \mu'_j r'_{j,i}.$$

Solution 9.3

Denote this Markov chain by $(X_n)_{n\in\mathbb{N}_0}$ and its state space by E. Since E is finite, we know that there exists at least one recurrent state $x\in E$. As $(X_n)_{n\in\mathbb{N}_0}$ is irreducible, all pairs of states communicate, so all states in E are recurrent. Suppose, for contradiction, that all states are null recurrent. Then we have,

$$\lim_{n\to\infty} P_y[X_n = x] = 0 \quad \forall x,y\in E.$$

We can take the sum over $x\in E$, and swap the limit and summation (since E is finite) to obtain

$$\lim_{n\to\infty}\sum_{x\in E} P_y[X_n = x] = 0 \quad \forall y\in E,$$

which is a contradiction, since $\sum_{x\in E} P_y[X_n = x] = P_y[X_n\in E] = 1$ for all n and y. Hence there exists a positive recurrent state, so all $x\in E$ are positive recurrent.

Solution 9.4

(a) The state $1\in\mathbb{N}$ is recurrent, as

$$\rho_{1,1} := P_1\Big[H_1<\infty\Big] = \mathbb{E}\left[\mathbf{1}(X_1=1)\,\mathbf{1}(H_1<\infty) + \mathbf{1}(X_1>1)\,\mathbf{1}(H_1<\infty)\right] = \sum_{i\in\mathbb{N}}\pi(i) = 1. \tag{1}$$

Moreover, in case the Markov chain jumps to state i starting from 1, then it will return to state 1 in exactly i steps. Hence $E_1[H_1] = \sum_{i\in\mathbb{N}} i\pi(i) < \infty$ by assumption, so state 1 is positive recurrent.

Define

$$m := \sup\{i\in\mathbb{N} : \pi(i) > 0\}$$

If $m=\infty$, then all states $i\in\mathbb{N}$ are connected with 1 and thus positive recurrent, by Theorem 3.16. If $m<\infty$, then all states $i\in\{1,2,\dots,m\}$ are connected with 1 and thus positive recurrent, by applying a result on irreducible homogeneous Markov chains from the course, with E restricted to $\{1,2,\dots,m\}$.

The states $i\in\mathbb{N}\setminus\{1,2,\dots,m\} =: F$ do not communicate as $i+1\to i$ for all $i\in F$, but not $i\to i+1$. The states $i\in F$ are transient, as for all $i\in F$ (defining $\rho_{i,j}$ similarly to (1)),

$$\rho_{i,i} = r_{i,i-1}\cdot\ldots\cdot r_{2,1}\rho_{1,i} = \rho_{1,i} = 0 < 1.$$

(b) Let X be an integer-valued random variable with distribution π. By assumption,

$$E[X] = \sum_{i\in\mathbb{N}} i\pi(i) < \infty.$$

We define a distribution $(\nu_i)_{i\in\mathbb{N}}$ by

$$\nu_i := \frac{P\left[X \geq i\right]}{E[X]}, \quad i \in \mathbb{N}.$$

To show that $(\nu_i)_{i\in\mathbb{N}}$ is a stationary distribution, we check $\nu_j = \sum_{i\in\mathbb{N}} \nu_i r_{i,j} \quad \forall j \in \mathbb{N}$:

$$\begin{aligned}
\sum_{i\in\mathbb{N}} \nu_i r_{i,j} &= \nu_1 r_{1,j} + \nu_{j+1} r_{j+1,j} \\
&= \frac{P\left[X \geq 1\right]}{E[X]} \pi\left(j\right) + \frac{P\left[X \geq j+1\right]}{E[X]} \\
&= \left(1 \cdot P\left[X = j\right] + P\left[X \geq j+1\right]\right) / E[X] \\
&= P\left[X \geq j\right] / E[X] \\
&= \nu_j.
\end{aligned}$$

The stationary distribution $(\nu_i)_{i\in\mathbb{N}}$ is reversible if and only if support $(\pi) = \{1, 2\}$, i.e. $\pi(1) + \pi(2) = 1$. If support $(\pi) \neq \{1, 2\}$, then there exists $i \geq 3$ with $\pi(i) > 0$. Thus, $\nu_1 r_{1,i} = \nu_1 \pi(i) > 0$, but $\nu_i r_{i,1} = 0$. Hence $\nu_1 r_{1,i} \neq \nu_i r_{i,1}$, and X is not reversible.

Remark. This example shows that a stationary distribution is not necessarily reversible. It also shows that one cannot skip the condition of irreducibility to have equivalence between existence of a stationary distribution and positive recurrence of all the states.

Solution 9.5

(a)

(i)+(ii) Both states are connected. As the state space is finite, both states are thus recurrent. However, we can prove directly that the states are recurrent. The state 0 is recurrent as

$$\begin{aligned}
\rho_{0\,0} &:= P_0[\textstyle\bigcup_{k=1}^{\infty}\{X_k = 0, X_{k-1} = 1, \ldots, X_1 = 1\}] \\
&= \textstyle\sum_{k=1}^{\infty} P_0[X_1 = 1, \ldots, X_{k-1} = 1, X_k = 0] \\
&= 1 - p + pr \textstyle\sum_{n=0}^{\infty} (1-r)^n \\
&= 1 - p + pr\frac{1}{r} \\
&= 1
\end{aligned}$$

Analogously, we can prove that the state 1 is recurrent.

(iii)

$$P = \begin{bmatrix} 1-p & p \\ r & 1-r \end{bmatrix} = TDT^{-1},$$

where

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1-p-r \end{bmatrix}, T = \begin{bmatrix} 1 & -p \\ 1 & r \end{bmatrix}, T^{-1} = \frac{1}{p+r} \begin{bmatrix} r & p \\ -1 & 1 \end{bmatrix}.$$

It follows

$$P^n = TD^nT^{-1} = T \begin{bmatrix} 1 & 0 \\ 0 & (1-p-r)^n \end{bmatrix} T^{-1}.$$

(iv)

$$\lim_{n\to\infty} P^n = T \lim_{n\to\infty} D^n T^{-1} = T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} T^{-1} = \frac{1}{p+r} \begin{bmatrix} r & p \\ r & p \end{bmatrix}.$$

b) (i) The states 0 and 3 are recurrent. The states 1 and 2 are transient.

(ii) The states 0 and 3 are not connected.

(iii)

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ q & 0 & p & 0 \\ 0 & p & 0 & q \\ 0 & 0 & 0 & 1 \end{bmatrix} = TDT^{-1},$$

with

$$D = \begin{bmatrix} p & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$T = \begin{bmatrix} 0 & 0 & -p & 1+p \\ 1 & -1 & 0 & 1 \\ 1 & 1 & 1-p & p \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; T^{-1} = \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 & -1 \\ \frac{1-p}{p+1} & -1 & 1 & \frac{p-1}{p+1} \\ 0 & 0 & 0 & 2 \\ \frac{2}{p+1} & 0 & 0 & \frac{2p}{p+1} \end{bmatrix}.$$

$$P^n = TD^nT^{-1} = T \begin{bmatrix} p^n & 0 & 0 & 0 \\ 0 & (-p)^n & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} T^{-1}$$

(iv)

$$\lim_{n\to\infty} P^n = T \lim_{n\to\infty} D^n T^{-1} = T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{1+p} & 0 & 0 & \frac{p}{1+p} \\ \frac{p}{1+p} & 0 & 0 & \frac{1}{1+p} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) (i) The state 3 is recurrent. The states 0,1 and 2 are transient.

(iii)

$$P = \begin{bmatrix} q & p & 0 & 0 \\ 0 & q & p & 0 \\ 0 & 0 & q & p \\ 0 & 0 & 0 & 1 \end{bmatrix} = T(D+N)T^{-1},$$

where

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-p & 0 & 0 \\ 0 & 0 & 1-p & 0 \\ 0 & 0 & 0 & 1-p \end{bmatrix}, \; N = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$T = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & \frac{1}{p} & 0 \\ 1 & 0 & 0 & \frac{1}{p^2} \\ 1 & 0 & 0 & 0 \end{bmatrix}, \; T^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & p & 0 & -p \\ 0 & 0 & p^2 & -p^2 \end{bmatrix}.$$

It follows

$$P^n = T(D+N)^nT^{-1},$$

with

$$(D+N)^n = \sum_{k=0}^{n} \binom{n}{k} D^{n-k} N^k.$$

(iv)

$$\lim_{n\to\infty} P^n = T \lim_{n\to\infty} (D+N)^n T^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$