ETH Zürich FS 2015
D-MATH
Prof. Antti Knowles

Coordinator
Thomas Cayé

Applied Stochastic Processes

Solution Sheet 11

Exercise 11.1

We have seen in the lecture that the non-explosion assumption is equivalent to

$$\sum_{n\geqslant 0} \lambda(X'_n)^{-1} = \infty, \quad \mathbb{P}'_x\text{-a.s. for all } x \in E.$$

(a)
$$\sum_{n\geqslant 0} \lambda(X'_n)^{-1} \geqslant \sum_{n\geqslant 0} c^{-1} = \infty.$$

(b) We have $\sup_{x\in E} \lambda(x) = c < \infty$, hence **(b)** follows from **(a)**.

(c) $\mathbb{P}'_x[\cap_{n\geqslant 0}\{X'_n \in \mathcal{T}\}] = 0$ implies that for $\mathbb{P}'_x$-a.a. ω there is $n_0(\omega) < \infty$ such that $X'_n(\omega) \in \mathcal{T}^c$ for all $n \geqslant n_0(\omega)$. This implies that $\mathbb{P}'_x$-a.s. there is a state $y \in T^c$, which the chain visits infinitely often. Define $N_x = \sum_{n=0}^{\infty} \mathbb{1}\,(X'_n = x)$. Then $\mathbb{P}'_x$-a.s. there exists $y \in T^c$, such that $N_y = \infty$. We have
$$\sum_{n\geqslant 0} \lambda(X'_n)^{-1} = \sum_{x\in E} \frac{N_x}{\lambda(x)},$$
hence the assumption $\lambda(y) < \infty$ implies the claim.

Take the Markov chain in continuous time on the state space $\mathbb{N}$ that starts at 0 $\mathbb{P}$-a.s., that has the following jump rate and transition probability

$$\begin{aligned} \lambda(x) =& (x+1)^2 \text{ for } x \in \mathbb{N}, \\ q_{x,y} =& \begin{cases} 1 & \text{if } y = x+1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We have then
$$\sum_{n\geqslant 0} \lambda(X'_n)^{-1} = \sum_{n\geqslant 1} \frac{1}{n^2} = \frac{\pi^2}{6} < \infty,$$
and this chain does not satisfy the non-explosion assumption.

Exercise 11.2

$R(t)$ ist the solution of the Kolmogorov backward equation (KBE)

$$R'(t) = \Lambda R(t)\,,\; t \geqslant 0\,, \quad R(0) = \text{id}$$

As the state space is finite, the unique solution to both the KBE and the Kolmogorov forward equation (KFE) is given by

$$R(t) = \exp(\Lambda t)\,,\; t \geqslant 0$$

Note that $\Lambda = BDB^{-1}$ with

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix} B = \begin{pmatrix} 1 & 1 & 7 \\ 1 & 5 & -11 \\ 1 & -3 & 1 \end{pmatrix},\; B^{-1} = \frac{1}{48}\begin{pmatrix} 14 & 11 & 23 \\ 6 & 3 & -9 \\ 4 & -2 & -2 \end{pmatrix}$$

Thus,

$$\begin{aligned} R(t) &= \exp(\Lambda t) \\ &= B\exp(Dt)B^{-1} \\ &= B\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-3t} \end{pmatrix} B^{-1} \\ &= \tfrac{1}{48}\begin{pmatrix} 14+6e^{-2t}+28e^{-3t} & 11+3e^{-2t}-14e^{-3t} & 23-9e^{-2t}-14e^{-3t} \\ 14+30e^{-2t}-44e^{-3t} & 11+15e^{-2t}+22e^{-3t} & 23-45e^{-2t}+22e^{-3t} \\ 14-18e^{-2t}+4e^{-3t} & 11-9e^{-2t}-2e^{-3t} & 23+27e^{-2t}-2e^{-3t} \end{pmatrix} \end{aligned}$$

Exercise 11.3

Note that we need to assume that X_t is integrable for all $t \geqslant 0$.

(a) We have $\lambda(i) = \lambda_i + \mu_i = (\lambda+\mu)i + a$. With $N_x = \sum_{n=0}^{\infty} \mathbb{1}(X'_n = x)$ we obtain

$$\sum_{n\geqslant 0} \lambda(X'_n)^{-1} = \sum_{i=0}^{\infty} \frac{N_i}{\lambda(i)} \geqslant \sum_{k=0}^{\infty} \frac{1}{(\lambda+\mu)i + a} = \infty,$$

and hence the non-explosion assumption holds.

(b) The forward Kolmogorov differential equations for a birth and death process are given by

$$\begin{aligned} &r'_{i,0}(t) = -\lambda_0 r_{i,0}(t) + \mu_1 r_{i,1}(t), \\ &r'_{i,j}(t) = \lambda_{j-1} r_{i,j-1}(t) - (\lambda_j + \mu_j) r_{i,j}(t) + \mu_{j+1} r_{i,j+1}(t),\; j \geqslant 1, \end{aligned}$$

and the boundary condition $r_{i,j}(0) = \delta_{ij}$. For linear growth with immigration these equations simplify to

$$\begin{aligned} &r'_{i,0}(t) = -a r_{i,0}(t) + \mu r_{i,1}(t), \\ &r'_{i,j}(t) = (\lambda(j-1)+a) r_{i,j-1}(t) - ((\lambda+\mu)j + a) r_{i,j}(t) + \mu(j+1) r_{i,j+1}(t),\; j \geqslant 1. \end{aligned}$$

We obtain, assuming absolute summability of the middle term uniformly in t on compact sets,

$$M'(t) = \sum_{j=1}^{\infty} j r'_{i,j}(t) = a \underbrace{\sum_{j=1}^{\infty} j(r_{i,j-1}(t) - r_{i,j}(t))}_{=1}$$
$$+ \lambda \underbrace{\sum_{j=1}^{\infty} j((j-1)r_{i,j-1}(t) - j r_{i,j}(t))}_{=M(t)}$$
$$+ \mu \underbrace{\sum_{j=1}^{\infty} j(-j r_{i,j}(t) + (j+1) r_{i,j+1}(t))}_{=-M(t)}$$
$$= a + (\lambda - \mu) M(t).$$

The initial condition is clear.

(c) The solution of the equation is given by

$$M(t) = at + i \quad \text{if } \mu = \lambda,$$

and

$$M(t) = \frac{a}{\lambda - \mu}(e^{(\lambda-\mu)t} - 1) + i e^{(\lambda-\mu)t} \quad \text{if } \lambda \neq \mu.$$

Exercise 11.4

(a) By the definition of the chain $(X'_n)_{n \geqslant 0}$, it is clear that all states of the discrete skeleton are connected, hence the chain is irreducible. We have as well that for $n \geqslant 2$

$$\mathbb{P}'_0[H'_0 = n] = q p^{n-2},$$

where we defined $H'_0 = \inf\{k \geqslant 1, X'_k = 0\}$. We then obtain $\mathbb{E}'_0[H'_0] = \sum_{n=2}^{\infty} n q p^{n-2} < \infty$, so that 0 and hence all $x \in \mathbb{N}$ are positive recurrent for $(X'_n)_{n \geqslant 0}$.

(b) For all $x \in E$, with probability 1 under $\mathbb{P}'_x$, we have

$$\sum_{n \geqslant 0} \lambda(X'_n)^{-1} \geqslant \lambda(y)^{-1} \sum_{n \geqslant 0} \mathbb{1}\left(X'_n = y\right) = \infty, \ \mathbb{P}'_x\text{-a.s.}$$

as $\lambda(y) > 0$ and all $y \in E$ are recurrent.

This implies that $(X_t)_{t \geqslant 0}$ is a pure jump process with no explosion for any jump rate function $\lambda(\cdot) : \mathbb{N} \to (0, \infty)$.

(c) We have

$$E_0[\tilde{H}_0] = E^{\mathbb{P}_0}\left[\sum_{n=1}^{H'_0} (S_n - S_{n-1})\right]$$
$$= E^{\mathbb{P}'_0}\left[\sum_{n=0}^{H'_0 - 1} \int_0^{\infty} u \lambda(X'_n) e^{-\lambda(X'_n) u} du\right]$$
$$= E^{\mathbb{P}'_0}\left[\sum_{n=0}^{H'_0 - 1} \lambda(X'_n)^{-1}\right],$$

and given the way the chain $(X'_n)_{n\geqslant 0}$ moves

$$= \frac{1}{\lambda(0)} + E^{\mathbb{P}'_0}\left[\sum_{n=1}^{H'_0-1} \lambda(n)^{-1}\right]$$
$$= \frac{1}{\lambda(0)} + \sum_{m=1}^{\infty} \frac{1}{\lambda(m)} \mathbb{P}'_0[H'_0 > m]$$
$$= \frac{1}{\lambda(0)} + \sum_{m=1}^{\infty} \frac{1}{\lambda(m)} p^{m-1}.$$

(d) If we choose $\lambda(x) = p^x$, it is immediate from **(c)** that $E_0[\tilde{H}_0] = \infty$, so $(X'_n)_{n\geqslant 0}$ is positive recurrent, but $E_0[\tilde{H}_0] = \infty$ (with 0 not absorbing) so 0 is not positive recurrent for $(X_t)_{t\geqslant 0}$.

Exercise 11.5

(a) Note that

$$q_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i}, \quad q_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}, \quad q_{i,j} = 0 \text{ for all } j \notin \{i-1, i+1\}$$

The generator matrix Λ is given by

$$\Lambda = \begin{pmatrix} -\lambda_0 & \lambda_0 & & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & & \\ & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

We have to solve $\pi^T\Lambda = 0$, $\pi = (\pi_i)_{i\in\mathbb{N}}$, with $\pi_0 = 1$. This is equivalent to

$$-\lambda_0 + \mu_1\pi_1 = 0$$
$$\lambda_i\pi_i - (\lambda_{i+1} + \mu_{i+1})\,\pi_{i+1} + \mu_{i+2}\pi_{i+2} = 0 \text{ for all } i \in \mathbb{N}$$

Summing up over $\{0, 1, \ldots, n-2\}$ yields

$$\sum_{i=0}^{n-2} \lambda_i\pi_i + \sum_{i=0}^{n} \mu_i\pi_i = \sum_{i=0}^{n-1} \lambda_i\pi_i + \sum_{i=0}^{n-1} \mu_i\pi_i$$

It follows

$$\mu_n\pi_n = \lambda_{n-1}\pi_{n-1}$$

and thus for $n \geq 1$:

$$\pi_n = \frac{\lambda_{n-1}}{\mu_n}\pi_{n-1} = \ldots = \frac{\prod_{i=0}^{n-1}\lambda_i}{\prod_{i=1}^{n}\mu_i}$$

A stationary distribution exists if and only if

$$\sum_{n=1}^{\infty} \pi_n = \sum_{n=1}^{\infty} \frac{\prod_{i=0}^{n-1}\lambda_i}{\prod_{i=1}^{n}\mu_i} < \infty$$

(b) For a stationary distribution $\nu = (\nu_i)_{i \in \mathbb{N}}$

$$\sum_{i=0}^{\infty} \nu_i = 1$$

must hold. Hence, the stationary distribution ν is given by

$$\nu_n = \frac{\pi_n}{1 + \sum_{n=1}^{\infty} \frac{\prod_{i=0}^{n-1} \lambda_i}{\prod_{i=1}^{n} \mu_i}}.$$