ETH Zürich FS 2015
D-MATH
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Brownian Motion and Stochastic Calculus
Exercise Sheet 7

1. For a function $f : [0, \infty) \to \mathbb{R}$, we define its variation $|f| : [0, \infty) \to [0, \infty]$ by

$$|f|(t) := \sup \left\{ \sum_{t_i \in \Pi} \big| f(t_{i+1}) - f(t_i) \big| \, \middle| \, \Pi \text{ is a partition of } [0, t] \right\}.$$

We say that f has finite variation (FV) if $|f|(t) < \infty$ for all $t \geq 0$.

(a) Show that f has finite variation if and only if there exist non decreasing functions $f_1, f_2 : [0, \infty) \to \mathbb{R}$ such that $f = f_1 - f_2$.
Hint: Show that $|f|$ is non decreasing.

Recall that if f is a non decreasing and continuous function, then there exists a unique positive measure μ_f on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ such that $\mu_f\big([0, t]\big) = f(t) - f(0)$ for all $t \geq 0$. Therefore, if f is non decreasing and continuous, we call a function $g : [0, \infty) \to \mathbb{R}$ *f-integrable in the Lebesgue–Stieltjes sense* if $\int_0^\infty |g(s)| \, \mu_f(ds) < \infty$. In that case, we define $\int g(s) \, df(s) := \int g(s) \, \mu_f(ds)$ and call it the *Lebesgue–Stieltjes integral*.

(b) Let f be of finite variation and continuous and $g : [0, \infty) \to \mathbb{R}$ such that $\int_0^\infty |g(s)| \, \mu_{|f|}(ds) < \infty$. Show that there are non decreasing, continuous functions $f_1, f_2 : [0, \infty) \to \mathbb{R}$ such that $f = f_1 - f_2$ and both

$$\int_0^\infty |g(s)| \, \mu_{f_1}(ds) < \infty, \qquad \int_0^\infty |g(s)| \, \mu_{f_2}(ds) < \infty.$$

Moreover, show that

$$\int g(s) \, df(s) := \int g(s) \, \mu_{f_1}(ds) - \int g(s) \, \mu_{f_2}(ds)$$

is well-defined.

Hint: Recall that if f has finite variation and continuous, then $|f|$ is continuous.

Remark: If f is of finite variation and continuous, we call g *f-integrable in the Lebesgue–Stieltjes sense* if g satisfies $\int_0^\infty |g(s)| \, \mu_{|f|}(ds) < \infty$.

Bitte wenden!

2. Assume we have a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ satisfying the usual conditions. Let $\mathcal{M}^c_{0,\text{loc}}$:= {the set of $(P, \mathbb{F})$-continuous local martingales starting in 0} and $\mathcal{H}^{2,c}_0$:={the set of continuous $(P, \mathbb{F})$- martingales $(M_t)_{t\geq 0}$ starting in 0 which are bounded in $L^2(P)$, i.e., $\sup_{t\geq 0} E[M_t^2] < \infty$}.

(a) Let $M \in \mathcal{M}^c_{0,\text{loc}}$. Prove that $M \in \mathcal{H}^{2,c}_0$ if and only if $E[\langle M \rangle_\infty] < \infty$.

(b) A stochastic process X is said to be *of class* (DL) if for all $a > 0$, the family

$$\mathfrak{X}_a := \{X_\tau | \tau \text{ stopping time }, \tau \leq a\ P\text{-a.s.}\}$$

is uniformly integrable. Show that a local martingale null at 0 is a (true) martingale null at 0 if and only if it is of class (DL).

Remark: If M is continuous local martingale, often the quadratic variation process of M is denoted by $\langle M \rangle$. If M is a local martingale which also admits jumps, then the quadratic variation process is denoted by $[M]$. So, if the process M is continuous, then $\langle M \rangle$ and $[M]$ coincide.

3. Let B be a Brownian motion in $\mathbb{R}^3$, $0 \neq x \in \mathbb{R}^3$ and define the process $M = (M_t)_{t\geq 0}$ by

$$M_t = \frac{1}{|x + B_t|}.$$

This is well defined as a 3-dimensional Brownian motion does not hit points, as seen in the lecture.

a) Show that M is a continuous local martingale.
Hint: Use Itô's formula.
Moreover, show that M is bounded in L^2, i.e., $\sup_{t\geq 0} E[|M_t|^2] < \infty$.
Hint: For any $t \geq 0$, show that

$$E\left[|M_t|^2 1_{\{|M_t| \geq \frac{2}{|x|}\}}\right] = (2\pi t)^{-\frac{3}{2}} \int_{|y| \leq \frac{|x|}{2}} \frac{1}{|y|^2} \exp\left(-\frac{|y - x|^2}{2t}\right) dy$$

and estimate the right-hand side from above using the reverse triangle inequality.

b) Show that M is a *strict local martingale*, i.e., M is not a martingale.
Hint: Show that $E[M_t] \to 0$ as $t \to \infty$. To this end, similarly to part **a)**, compute $E[M_t]$ and use the reverse triangle inequality as a first estimate. Then compute the resulting integral using spherical coordinates.

Remark: This is the standard example of a local martingale which is not a (true) martingale. It also shows that even good integrability properties like boundedness in L^2 are not enough to guarantee the martingale property.

Siehe nächstes Blatt!

4. Matlab Exercise Let $x = (1,1,1)^T \in \mathbb{R}^3$. We consider the first time that a three-dimensional Brownian motion starting in x hits the unit ball $B_1(0)$, i.e.,

$$T_{B_1(0)} := \inf\{t > 0 | x + B_t \in B_1(0)\},$$

where B is a standard Brownian motion starting in $(0,0,0)^T \in R^3$. From the lecture we know that $P[T_{B_1(0)} < \infty] = 1/\|x\|$. The goal of this exercise is to compute this probability numerically. That is, take $T = 200$ and simulate 10^4 sample paths of a three dimensional Brownian motion ($dt = 10^{-2}$). For each sample path determine whether the Brownian motion hits the unit ball and compute $P[T_{B_1(0)} < \infty]$ numerically.

Hint: Use Monte-Carlo simulation to compute $E[\mathbf{1}_{\{T_{B_1(0)}<\infty\}}]$. The essential idea of Monte Carlo simulation is that – by the law of large numbers – for large $m \in \mathbb{N}$ and an i.i.d. sequence $X_1, \ldots, X_m$ we have

$$E[X_1] \approx \frac{1}{m} \sum_{k=1}^{m} X_k.$$