ETH Zürich FS 2015
D-MATH
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Brownian Motion and Stochastic Calculus
Exercise Sheet 11

1. Let $(B_t)_{t\geq 0}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. Consider the SDE

$$dX_t = \Big(\sqrt{1+X_t^2} + \frac{1}{2}X_t\Big)\,dt \;+\; \sqrt{1+X_t^2}\,dB_t, \quad X_0 = x \in \mathbb{R}. \tag{1}$$

a) Show that for any $x \in \mathbb{R}$ the SDE defined in (1) has a unique strong solution.
Hint: Verify that the coefficients of the SDE satisfy the required Lipschitz and linear growth condition.

b) Show that $(X_t)_{t\geq 0}$ defined by $X_t = \sinh\big(\text{arsinh}\, x + t + B_t\big)$ is the unique solution of (1).
Hint: Consider the process $(Y_t)_{t\geq 0}$ defined by $Y_t := \text{arsinh}\, B_t$.

2. Let $W = (W_t)_{t\geq 0}$ be a Brownian motion defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ satisfying the usual conditions.

a) Consider the *Ornstein-Uhlenbeck process*

$$X_t = xe^{-\lambda t} + \nu(1 - e^{-\lambda t}) + \int_0^t \sigma e^{\lambda(s-t)} dW_s, \quad t \geq 0 \tag{2}$$

for an $x \in \mathbb{R}$, where ν and $\lambda, \sigma > 0$ are real constants. Show that X satisfies the Ornstein-Uhlenbeck SDE:

$$dX_t = \lambda(\nu - X_t)dt + \sigma dW_t, \quad X_0 = x.$$

Hint: Apply Itô's formula to $f(x,t) = xe^{\lambda t}$.

b) Calculate the mean and variance functions of X:

$$T \mapsto \mathbb{E}[X_T], \quad \text{and} \quad T \mapsto \text{Var}[X_T].$$

Bitte wenden!

c) Let now $\nu = 0$. Show that there is a Brownian motion B such that $Y := X^2$ satisfies the following Cox-Ingersoll-Ross SDE

$$dY_t = (-2\lambda Y_t + \sigma^2)\, dt + 2\sigma \sqrt{Y_t}\, dB_t.$$

3. Let $W = (W_t)_{t\geq 0}$ be a Brownian motion defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ satisfying the usual conditions. Assume that the filtration $\mathbb{F}$ is generated by the Brownian motion W. Consider the Tanaka SDE

$$dX_t = sgn(X_t) dW_t, \quad X_0 = 0,$$

where $sgn(x)$ denotes the sign function, i.e., $sgn(x) = 1$ if $x > 0$ and $sgn(x) = -1$ if $x \leq 0$.

a) Show that the Tanaka SDE has no strong solution.
Hint:

- Assume there exists a strong solution and derive a contradiction.
- You can use the following result (Tanaka's formula): Let X be a continuous semimartingale. There exists a continuous increasing adapted process $(L_t)_{t\geq 0}$ such that
$$|X_t| - |X_0| = \int_0^t sgn(X_s) dX_s + L_t.$$
Moreover, it can be shown that L is $\mathbb{F}(|X|)$ adapted, where $\mathbb{F}(|X|)$ denotes the filtration generated by $|X|$.

b) Show that the SDE admits a weak solution.

4. Matlab Exercise Given a finite time horizon $T = 1$, the aim of this exercise is to simulate the Ornstein-Uhlenbeck process and the Cox-Ingersoll-Ross process from (Ex 11-2) on the time interval $[0, T]$ using the *Euler-Maruyama scheme*.[1]
To this end, let W be a dimensional Brownian motion. We define an equidistant decomposition $\{0 = t_0 < \ldots < t_n = T\}$ of the interval $[0, T]$ by setting

$$t_i := \frac{i}{M} T, \quad i = 0, \ldots, M = 10^3.$$

If X is a process on the interval $[0, T]$ satisfying the stochastic differential equation

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t$$

[1] This is the stochastic version of the Euler-scheme for ODEs.

Siehe nächstes Blatt!

with initial condition $X_0 = x$ for an $x \in \mathbb{R}$, and $t_0 = 0 < t_1 < \ldots < t_M = T$ is a given discretization of the time interval $[0, T]$, then an *Euler-Maruyama approximation*[2] of X is given by the iterative scheme: $X_0 = x$ and

$$X_{t_{i+1}} = X_{t_i} + a(t_i, X_{t_i})(t_{i+1} - t_i) + b(t_i, X_{t_i})(W_{t_{i+1}} - W_{t_i}), \quad i = 0, \ldots, M-1.$$

a) Simulate 10 sample paths of the OU-process X from Ex 11-2 a) with $\lambda = 1$, $\nu = 1.2$, $\sigma = 0.3$ and $X_0 = 1$.

b) Use Monte-Carlo simulation ($N = 10^5$) to compute $\mathbb{E}[X_1], \mathbb{E}[X_1^2], \mathbb{E}[X_1^+]$.

c) Consider the *Cox-Ingersoll-Ross* process Y defined by the following SDE:

$$dY_t = \lambda(\nu - Y_t)dt + \sigma\sqrt{Y_t}dW_t, \quad Y_0 = y.$$

Repeat the tasks (a) and (b) for the CIR process. Is there a *potential* problem for the simulation procedure?

Remark: For part c) it can be shown that under the parameter restriction $2\lambda\nu \geq \sigma^2$ the process Y is P. a.s. strictly positive.

[2] As a reference for the Euler-Maruyama approximation see for example Section 3.2 of *Numerical Solution of SDE Through Computer Experiments* (Kloeden, Platen, Schurz).