Brownian Motion and Stochastic Calculus
Exercise Sheet 12

1. Consider the SDE

$$dX_t^x = a(X_t^x)\,dt + b(X_t^x)\,dW_t, \quad X_0^x = x,$$

where W is an $\mathbb{R}^m$-valued Brownian motion, $a : \mathbb{R}^d \to \mathbb{R}^d$ and $b : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are measurable and locally bounded. Assume that for any $x \in U$, we have for $T_U^x := \inf\{s \geq 0; X_s^x \notin U\}$ that T_U^x is P-integrable. Moreover, consider the boundary problem

$$\begin{cases} Lu(x) + c(x)u(x) &= -f(x) \quad \text{for } x \in U, \\ u(x) &= g(x) \quad \text{for } x \in \partial U, \end{cases}$$

where $U \neq \emptyset$ is a bounded open subset of $\mathbb{R}^d$, $f \in C_b(U)$, $g \in C_b(\partial U)$, $c \leq 0$ is a uniformly bounded function on $\mathbb{R}^d$ and L is defined by

$$Lf(x) := \sum_{i=1}^d a_i(x)\,\frac{\partial f}{\partial x^i}(x) + \frac{1}{2}\sum_{i,j=1}^d \left(b\,b^{tr}\right)_{ij}(x)\,\frac{\partial^2 f}{\partial x^i\,\partial x^j}(x).$$

Show that if $u \in C^2(U) \cap C(\bar{U})$ is a solution of the above boundary problem and $(X_t^x)_{t\geq 0}$ is a solution of the SDE for some $x \in U$, then

$$u(x) = E\left[g(X_{T_U^x}^x)\,\exp\Big(\int_0^{T_U^x} c(X_s^x)\,ds\Big)\right] + E\left[\int_0^{T_U^x} f(X_s^x)\,\exp\Big(\int_0^s c(X_r^x)\,dr\Big)\,ds\right].$$

2. Let X be a Lévy process in $\mathbb{R}^d$ and $f_t(u) = E[e^{i\,u^{\mathrm{tr}}X_t}]$.

a) Show that X is stochastically continuous, i.e., $t \mapsto X_t$ is continuous in probability, i.e., for all $\varepsilon > 0$ and $t \geq 0$ we have $P[|X_t - X_s| > \varepsilon] \to 0$ as $s \to t$.

b) Show that $f_{t+s}(u) = f_t(u)f_s(u)$ for all $s, t \geq 0$ and $f_0(u) = 1$ for any $u \in \mathbb{R}^d$.

c) Use **b)** to show that $f_r(u) = f_1(u)^r$ for all *rational* $r \geq 0$.

d) Show that $t \mapsto f_t(u)$ is right-continuous and conclude that $f_t(u) = f_1(u)^t$ and that $f_t(u) \neq 0$ for all $t \geq 0$ and $u \in \mathbb{R}^d$.

e) Let $d = 1$. If $E[|X_1|] < \infty$, then $E[X_t] = tE[X_1]$ for all $t \geq 0$.

3. **a)** Let N be a one-dimensional Poisson process and $(Y_i)_{i\geq 1}$ i.i.d. $\mathbb{R}^d$-valued random variables independent of N. We define the *compound Poisson process* by $X_t := \sum_{i=1}^{N_t} Y_i$. Show that X is a Lévy process and calculate its Lévy triplet.

b) Is there a Lévy process X such that X_1 is uniformly distributed on $[0, 1]$?

c) Let X and Y be both Lévy processes with respect to a filtration $(\mathcal{F}_t)$. Show that if $E[e^{i\,u^{\mathrm{tr}}X_t}e^{i\,v^{\mathrm{tr}}Y_t}] = E[e^{i\,u^{\mathrm{tr}}X_t}]\,E[e^{i\,v^{\mathrm{tr}}Y_t}]$ for all $u, v \in \mathbb{R}^d$ and $t \geq 0$, then X and Y are independent.

4. **Matlab Exercise** Let $T = 1$, $N = (N_t)_{t\geq 0}$ a $(P, \mathcal{F})$ Poisson process with intensity parameter $\lambda = 2$ and X be a compound Poisson process, i.e.,

$$X_t = \sum_{i=1}^{N_t} Z_i$$

with i.i.d random variables Z_i satisfy

$$Z_i = \begin{cases} 1, & p = 0.5, \\ -1, & p = 0.5. \end{cases}$$

(i) Simulate 10 sample paths of the processes N and X using an equidistant time grid, i.e., $t_i = T \cdot i/M, i = 0, \cdots, M = 10^3$.
Hint: Use the fact that the increments of N is Poisson distributed.

(ii) Compute $\mathbb{E}[e^{N_T}]$, $\mathbb{E}[e^{X_T}]$ using Monte-Carlo simulation with $N = 10^5$ sample paths and compare it with the theoretical values.