ETH Zürich FS 2015
D-MATH
Prof. Dr. Josef Teichmann

Coordinator
Ren Liu

Brownian Motion and Stochastic Calculus
Exercise Sheet 2

1. A continuous time stochastic process is called a *Brownian Bridge* if it is a Gaussian process with mean 0 and covariance function $s(1-t)$, $s<t$. Let W be a Brownian motion and consider the process $X=(X_t)_{0\le t\le 1}$ defined by $X_t = W_t - tW_1$.

a) Show that X is a Brownian Bridge.
Hint: If $(X_{t_1},\ldots,X_{t_n})$ is the image of a linear transformation of another Gaussian vector, then $(X_{t_1},\ldots,X_{t_n})$ is also a Gaussian vector.

b) Show that X does **not** have independent increments.

Hint: Compute the covariance of the increments.

2. Show that if $(X_n)_{n\in\mathbb{N}}$ is a Gaussian process indexed by $\mathbb{N}$ and converges in probability to a random variable X as n goes to infinity, then it converges also in L^2 to X.

Hint:

(1) Show that $(X_n - X)_{n\in\mathbb{N}}$ is a sequence of normal random variables.

(2) To show (1) you can use the following result: Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables with $X_n \sim \mathcal{N}(\mu_n, \sigma_n^2)$ for each $n\in\mathbb{N}$. If the sequence $(X_n)_{n\in\mathbb{N}}$ converges in distribution to a random variable X, then the limits $\mu := \lim_{n\to\infty}\mu_n$ and $\sigma^2 := \lim_{n\to\infty}\sigma_n^2$ exist and $X\sim\mathcal{N}(\mu,\sigma^2)$.

3. Let $(S_i,\mathcal{S}_i)_{i\in\mathbb{N}_0}$ be a sequence of measurable spaces and consider the path space $\Omega = \times_{i=0}^{\infty} S_i := \{\omega = \{(x_0,x_1,\ldots)|x_i\in S_i\}$ with the canonical coordinate map $X_n(\omega) := x_n$. For each $n\ge 0$ let $\mathcal{F}_n = \sigma(X_0,\ldots,X_n)$ denote the sigma algebra of observable events up to time n, and $\mathcal{F} = \sigma(X_0,X_1,\ldots) = \sigma\left(\cup_{n=0}^{\infty}\mathcal{F}_n\right)$. Moreover, we assume that there exists

- an initial distribution P_0 on S_0

Bitte wenden!

- a sequence of stochastic kernels K_n from $(\times_{i=0}^{n-1} S_i, \mathcal{F}_{n-1})$ to $(S_n, \mathcal{S}_n)$.

Put differently, we define by induction,

- $Q_0 = P_0$ on $(S_0, \mathcal{S}_0)$
- $Q_{n+1} = Q_n \times K_{n+1}$ on $(\times_{i=0}^{n+1} S_i, \mathcal{F}_{n+1})$, for $n \geq 0$.

Then, the powerful **Ionescu-Tulcea** Theorem states that there exists a unique probabilty measure Q on $(\Omega, \mathcal{F})$ such that for all $n \geq 0$ we have

$$\pi_n \circ Q = Q_n,$$

where $\pi_n : \Omega \mapsto \times_{i=0}^{n} S_i$ with $\pi_n(\omega) = (x_0, \ldots, x_n)$ denotes the projection map onto $\times_{i=0}^{n} S_i$. That is, there exists a unique distribution Q on $(\Omega, \mathcal{F})$ such that for each $n \geq 0$ it behaves like Q_n on $(\times_{i=0}^{n} S_i, \mathcal{F}_n)$.
Let P_n be a sequence of probability distributions on $(S_i, \mathcal{S}_i)_{n\geq 1}$. For the special kernel $K_n((x_0, \ldots, x_{n-1}), \cdot) \equiv P_n$, the measure constructed in the Ionescu-Tulcea Theorem is called the *product measure* of the distributions P_n and is denoted by $\bigotimes_{n=0}^{\infty} P_n$.
Show that P is the product measure of the marginal distributions $P_n = P \circ X_n^{-1}$ if and only if the canonical coordinate maps $(X_n)_{n=0,1,\ldots}$ are independent with respect to P.

Hint: What is $Q_{n+1} = Q_n \times K_{n+1}$ if $K_n((x_0, \ldots, x_{n-1}), \cdot) \equiv P_n$?

4. **Matlab Exercise** Let $W = (W_t)_{0\leq t\leq 1}$ be a standard Brownian motion and $X_t = 1 + 2t + 2W_t$ denote a drifted Brownian motion. Simulate $N = 10$ sample paths of the processes X using the normal increment property of the Brownian motion. For this exercise you can use an equidistant time grid, i.e., $t_i = i/M, i = 0, \cdots, M = 10^3$.

   Hint:

   - Recall that $(W_{t_{i+1}} - W_{t_i}) \sim \mathcal{N}(0, t_{i+1} - t_i)$. Therefore, first simulate independent normal random variables and obtain the sample path of W by summing it up efficiently.
   - Make sure that your Brownian motion starts at 0.
   - The MATLAB-commands *randn*, *cumsum* and *repmat* might be useful.