ETH Zürich FS 2015
D-MATH
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Brownian Motion and Stochastic Calculus
Exercise Sheet 4

1. A function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is called locally Hölder continuous of order α at $x \in D$ if there exists $\delta > 0$ and $C > 0$ such that $|f(x) - f(y)| \leq C|x-y|^\alpha$ for all $y \in D$ with $|x-y| \leq \delta$. A function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is called locally Hölder continuous of order α, if it is locally Hölder continuous of order α at each $x \in D$.

a) Let $Z \sim N(0,1)$. Prove that $P[|Z| \leq \varepsilon] \leq \varepsilon$ for any $\varepsilon \geq 0$.

b) Prove that for any $\alpha > \frac{1}{2}$, P-almost all Brownian paths are nowhere on $[0,1]$ locally Hölder-continuous of order α.
Hint:

- Take any $M \in \mathbb{N}$ satisfying $M(\alpha - \frac{1}{2}) > 1$ and show that the set $\{W_\cdot(\omega)$ is locally α-Hölder at some $s \in [0,1]\}$ is contained in the set $B := \bigcup_{C\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \bigcap_{n\geq m} \bigcup_{k=0,\dots,n-M} \bigcap_{j=1}^{M} \left\{ |W_{\frac{k+j}{n}}(\omega) - W_{\frac{k+j-1}{n}}(\omega)| \leq C\frac{1}{n^\alpha} \right\}$.
- Show $P[B] = 0$.

c) The *Kolmogorov–Čentsov theorem* states that an $\mathbb{R}$-valued process X on $[0,T]$ satisfying

$$E\big[|X_t - X_s|^\gamma\big] \leq C\,|t-s|^{1+\beta}, \quad s,t \in [0,T],$$

where $\gamma, \beta, C > 0$, has a version which is locally Hölder-continuous of order α for all $\alpha < \beta/\gamma$. Use this to deduce that Brownian motion has for every $\alpha < 1/2$ a version which is locally Hölder-continuous of order α.

Remark: One can also show that the Brownian paths are *not* locally Hölder-continuous of order $1/2$. The exact modulus of continuity was found by P. Lévy.

2. Let $(W_t)_{t\geq 0}$ be a Brownian motion. For any $a > 0$ consider the $\mathbb{H}$- stopping times

$$T_a := \inf\big\{t > 0 \,\big|\, W_t \geq a\big\}, \quad \overline{T}_a := \inf\big\{t > 0 \,\big|\, |W_t| \geq a\big\},$$

where $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$ with $\mathcal{H}_t := \bigcap_{\varepsilon>0} \mathcal{H}^0_{t+\varepsilon} = \bigcap_{\varepsilon>0} \sigma(W_s, s \leq t)$.

Bitte wenden!

a) Show that the Laplace transform of T_a has the value

$$E\big[\exp(-\mu T_a)\big] = \exp\big(-a\sqrt{2\mu}\big), \quad \forall \mu > 0,$$

and show that $P[T_a < \infty] = 1$.
Hint: Consider the martingale $M_t^\lambda = \exp\left(\lambda W_t - \frac{\lambda^2}{2}t\right)$ and use the stopping theorem.

b) Show that the Laplace transform of $\overline{T}_a$ has the value

$$E\big[\exp(-\mu \overline{T}_a)\big] = \frac{1}{\cosh(a\sqrt{2\mu})}, \quad \forall \mu > 0.$$

Hint: Consider a martingale N^λ similar to M^λ and use the same approach as in **a)**.

3. Let W be a Brownian motion on $[0, \infty)$ and $S_0 > 0$, $\sigma > 0$, $\mu \in \mathbb{R}$ constants. The stochastic process $S = (S_t)_{t \geq 0}$ given by

$$S_t = S_0 \exp\big(\sigma W_t + (\mu - \sigma^2/2)t\big)$$

is called *geometric Brownian motion*.

(a) Prove that for $\mu \neq \sigma^2/2$, we have

$$\lim_{t\to\infty} S_t = +\infty \quad P\text{-a.s.} \quad \text{or} \quad \lim_{t\to\infty} S_t = 0 \quad P\text{-a.s.}$$

When do the respective cases arise?

(b) Discuss the behaviour of S_t as $t \to \infty$ in the case $\mu = \sigma^2/2$.

(c) For $\mu = 0$, show that S is a martingale, but not uniformly integrable.

Hint:

- Recall the strong law of large numbers for Brownian motion (cf. Corollary (1.2) in section 2.1 of the lecture notes): For a Brownian motion W

$$\lim_{t\to\infty} \frac{W_t}{t} = 0 \quad \text{P-a.s.}$$

4. Matlab Exercise The aim of the exercise is to simulate Brownian motion by recursively refining an initial sample path (simulated on a coarse grid). Let $T = 1$. Use the normal increment property of Brownian motion to simulate one sample path with time grid $dt = 1/10$ and $t_j = j \cdot dt$ for $j = 0, 1, \ldots, 10$ (cf. Ex 2-4). Refine this path by simulating the conditional values at the midpoints $(t_j + t_{j+1})/2$, $j = 0, \ldots, 9$ of all intervals (t_j, t_{j+1}). Repeat the process until the length of each of the subintervals become $1/(10 \cdot 2^8)$. Plot the path after each round of refinement.

Hint:

- Given the values of the Brownian motion at t_j and t_{j+1}, what is the distribution of $B_{\frac{t_j+t_{j+1}}{2}}$?
- The MATLAB command *subplot* might be useful.