ETH Zürich FS 2015
D-MATH
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Brownian Motion and Stochastic Calculus
Exercise Sheet 5

1. Let $(B_t)_{t\geq 0}$ be a Brownian motion and $M_t = sup_{s\leq t} B_s$. Show that the joint distribution of the pair (B_t, M_t) is given by

$$P\left(B_t \in dx, M_t \in dy\right) = \frac{2(2y-x)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2y-x)^2}{2t}\right) \mathbb{1}_{\{y\geq 0\}} \mathbb{1}_{\{x\leq y\}} dxdy.$$

Hint: Show that

(i) for $y > 0, x \leq y$, $\quad P(B_t \leq x, M_t \geq y) = P(B_t \geq 2y - x)$, this property is called 'reflection principle' for Brownian motion. To prove (1), let $T_y := \inf\{t > 0 | B_t \geq y\}$ and use the strong Markov property to compute $P(B_t - B_{T_y} \leq x - y, T_y \leq t)$.

(ii) for $y > 0, x \leq y$, $\quad P(B_t \leq x, M_t \leq y) = \Phi\left(\frac{x}{\sqrt{t}}\right) - \Phi\left(\frac{x-2y}{\sqrt{t}}\right)$,

(iii) for $y > 0, x \geq y$, $\quad P(B_t \leq x, M_t \leq y) = P(M_t \leq y) = \Phi\left(\frac{y}{\sqrt{t}}\right) - \Phi\left(-\frac{y}{\sqrt{t}}\right)$,
and for $y \leq 0$, $\quad P(B_t \leq x, M_t \leq y) = 0$.

2. Let $(X_t)_{t\geq 0}$ be the canonical one-dimensional Brownian motion and W_0 the Wiener measure (starting from 0). Let $S = \sup\left\{0 \leq u \leq 1 \,\middle|\, X_u = 0\right\} \vee 0$ be the time of the last zero before time 1. Show that

$$W_0\Big(S \leq s\Big) = \frac{1}{\pi}\int_0^s \frac{dv}{\sqrt{v(1-v)}} \left(= \frac{2}{\pi}\arcsin\sqrt{s}\right), \quad \text{for } 0 \leq s \leq 1.$$

Hint:

- Use that $\{S \leq s\} = \{H_0 \circ \vartheta_s > 1 - s\}$ where $H_0 := \inf\{s > 0 \,|\, X_s = 0\}$ is the first passage time through 0 and ϑ_s is the shift operator such that $\vartheta_t\omega(\cdot) = \omega(t+\cdot)$ for functions $\omega : [0, \infty) \to \mathbb{R}$. Intuitively, this means that if the time of the last zero is smaller than s, then the first passage through 0 after time s can only happen after at least $1 - s$ units of time have elapsed.

Bitte wenden!

- Employ the simple Markov property of Brownian motion
- Use Ex 5-1 to compute the law of H_x under W_0 for $x \neq 0$ (What is the law of H_0 under W_x)?

3. Consider a probability space $(\Omega, \mathcal{F}, P)$. For any sub-σ-algebra $\mathcal{G} \subset \mathcal{F}$, we let

$$\mathcal{N}(\mathcal{G}) = \{A \subset \Omega : \exists B \in \mathcal{G} \text{ with } A \subset B \text{ and } P[B] = 0\}$$

denote the collection of all subsets of P-nullsets in $\mathcal{G}$. For any filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ of $\mathcal{F}$, we define its *(P-) completion* $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{t\geq 0}$ by $\overline{\mathcal{F}}_t = \mathcal{F}_t \vee \mathcal{N}(\mathcal{F}_t)$, and its *(P-) augmentation* $\widetilde{\mathbb{F}} = (\widetilde{\mathcal{F}}_t)_{t\geq 0}$ by $\widetilde{\mathcal{F}}_t = \mathcal{F}_t \vee \mathcal{N}(\mathcal{F}_\infty)$ where $\mathcal{F}_\infty = \bigvee_{t\geq 0} \mathcal{F}_t$. Clearly,

$$\mathbb{F} \subset \overline{\mathbb{F}} \subset \widetilde{\mathbb{F}}. \tag{$\star$}$$

Show that if $\mathbb{F}$ is the (raw) filtration generated by the Brownian motion B realised on the canonical space $C[0, \infty)$, then both inclusions in ($\star$) are strict.

Hint:

- For the first inclusion, you can assume the existence of a non-Borel subset of $\mathbb{R}$ which does not contain 0.
- For the second claim, think of an event that has probability 0 but cannot be observed at time 0. Also recall that $A \in \overline{\mathcal{F}}_t$ if and only if there are two sets $F, G \in \mathcal{F}_t$ such that $F \subset A \subset G$ and $P[F] = P[G]$.

4. Matlab Exercise Verify numerically the arcsin law of the last visit time of the Brownian motion. That is, first simulate 10^4 Brownian sample paths on $[0, 1]$ using an equidistant time grid with 10^4 points, i.e., $t_i = i/M, i = 0, \dots, M = 10^4$. Then, compute the last visit time of each sample path and plot the empirical cumulative distribution function and compare it with its theoretical counterpart from Exercise 5-2.

Hint:

- To find a numerical zero you might want to round the numbers to the second decimal place.
- The MATLAB command *ecdf* might be useful.