ETH Zürich FS 2015
D-MATH
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Brownian Motion and Stochastic Calculus
Exercise Sheet 6

1. Let $(B_t)_{t\geq 0}$ be a Brownian motion and define the process $(M_t)_{t\geq 0}$ by $M_t = \sup_{0\leq s\leq t} B_s$. Show that for any fixed $t \geq 0$

$$M_t - B_t \overset{Law}{=} |B_t| \overset{Law}{=} M_t. \tag{1}$$

That is, show that the random variables have the same density functions.
Hint:

- We have already computed the joint law of (B_t, M_t) in Ex 5-1.

2. Let $(B_t)_{t\geq 0}$ be a Brownian motion and denote by $\mathcal{G}_t := \sigma(B_u,\, u \leq t),\ t \geq 0$. Define $\widetilde{R}_0 f(x) = f(x)$ and

$$\widetilde{R}_t f(x) = \frac{1}{\sqrt{2\pi t}} \int_0^\infty f(y) \left[\exp\left(-\frac{1}{2t}(y-x)^2\right) + \exp\left(-\frac{1}{2t}(y+x)^2\right)\right] dy, \quad t > 0.$$

Let us consider the process $(X_t)_{t\geq 0}$ by $X_t := |B_t|$. Show that

$$E\big[f(X_{t+h}) \,\big|\, \mathcal{G}_t\big] = \widetilde{R}_h f(X_t) \ \ P\text{-a.s. for } \ f \in b\mathcal{B}(\mathbb{R}) \text{ and } t, h \geq 0.$$

3. Let $(B_t)_{t\in[0,1]}$ be a Brownian motion on $(\Omega, \mathcal{F}, P)$ and define the process $(M_t)_{t\geq 0}$ by $M_t = \sup_{0\leq s\leq t} B_s$. Consider the random variable

$$D = \sup_{0\leq t'\leq 1} \big(\sup_{0\leq t\leq t'} B_t - B_{t'} \big). \tag{2}$$

That is, D characterizes the maximal possible "downfall" in trajectories of the Brownian motion on the time interval $[0, 1]$.

(a) Show that $D \overset{law}{=} \sup_{0\leq t\leq 1} |B_t|$.
Hint: You can use a stronger version of Ex 5-1, which is known as "Lévy's Theorem": The processes $M - B$ and $|B|$ have the saw law under P.

Bitte wenden!

(b) Show that $\sup_{0\le t\le 1}|B_t| \overset{law}{=} 1/\sqrt{\bar{T}_1}$, where $\bar{T}_1 = \inf\{t > 0 : |B_t| \ge 1\}$.
Hint: Rewrite $P[\sup_{0\le t\le 1}|B_t| \le x]$ using the self-similarity property of Brownian motion (cf. Proposition 1.1 (3) in Section 2.1 of the lecture notes).

(c) Conclude that $E[D] = \sqrt{\pi/2}$.
Hint: For $\sigma > 0$ use the identity

$$\sqrt{2/\pi}\int_0^\infty e^{-x^2/(2\sigma^2)}dx = \sigma,$$

to rewrite the expectation and apply the Laplace transform of $\bar{T}_1$ (cf. Ex 4-2) to conclude the result.

4. Matlab Exercise Verify the claim in Ex 6-1 numerically. That is, let $T = 1$, first simulate 10^4 Brownian sample paths on $[0, 1]$ using an equidistant time grid with 10^4 points, i.e., $t_i = i/M, i = 0, \ldots, M = 10^4$. Then, compute numerically the random variables $X_1^1 := M_1 - B_1$, $X_1^2 := |B_1|$ and $X_1^3 := M_1$ and plot the corresponding empirical cumulative distribution functions. Compare the ecdfs with its theoretical counterpart from Ex 5-1.

Hint:

- This exercise is very similar to Ex 5-4. Try to understand the sample solution of Ex 5-4 and adjust the code accordingly.
- The MATLAB command *repmat* might be useful.