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Brownian Motion and Stochastic Calculus
Exercise Sheet 8

1. For any $M \in \mathcal{M}^c_{0,\text{loc}}$, define as usual $M^*_t := \sup_{0\le s\le t} |M_s|$ for $t \ge 0$. Prove that for any $t \ge 0$ and $C, K > 0$, we have

$$P\big[M^*_t > C\big] \le \frac{4K}{C^2} + P\big[\langle M\rangle_t > K\big].$$

Hint: Find a stopping time σ_K such that the stopped process $M^{\sigma_K} \in \mathcal{H}^{2,c}_0$ and use the Tchebycheff and Doob inequalities.

Remark: Intuitively, this means that one can control the running supremum of M in terms of the quadratic variation of M.

2. Let $(\Omega, \mathcal{G}, \mathcal{G}_t, P)$ satisfying the usual conditions.

a) Show that every continuous *bounded* local martingale is a martingale.

b) Let $0 < T < \infty$ be a deterministic time. Show that any nonnegative continuous local martingale $(X_t)_{t\in[0,T]}$ with $E[X_0] < \infty$ is also a supermartingale, and if

$$E[X_T] = E[X_0],$$

then $(X_t)_{t\in[0,T]}$ is a martingale.

c) Show that for a continuous local martingale $(M_t)_{t\ge0}$ with M_0 bounded, one can find a sequence of stopping times $(S_n)_{n\in\mathbb{N}}$, P-a.s. tending to infinity, such that for each n, the stopped process $M^{S_n} := (M_{S_n\wedge t})_{t\ge0}$ is a bounded continuous martingale.

Hint: Use the fact that if $(M_t)_{t\ge0}$ is a right-continuous martingale and τ any stopping time, then the stopped process $M^\tau := (M_{\tau\wedge t})_{t\ge0}$ is a martingale.

d) Show that in general a process can be a martingale without being locally square integrable, i.e., find a process M and the underlying filtered probability space $(\Omega, \mathcal{F}, Q)$ such that X is a $(\mathcal{F}, Q)$ martingale but is not locally square integrable.

Hint: Think of a process which does not have square integrable jumps.

3. For $M \in \mathcal{M}^2_{0,loc}$, i.e., M is continuous local martingale starting in 0, $L^2_{loc}(M)$ is the space of all predictable processes for which there is a sequence of stopping times $\tau_n \uparrow \infty$ P- a.s. and such that for each n

$$\mathbb{E}\left[\int_0^{\tau_n} H_s^2 d\langle M\rangle_s\right] < \infty.$$

a) Show that for every $p \in (0, \infty)$, there are constants $c_p, C_p > 0$ only depending on p such that for every stopping time τ, every $M \in \mathcal{M}^c_{0,\text{loc}}$ and every locally bounded, predictable process H, we have

$$c_p E\left[\left(\int_0^\tau H_s^2\, d\langle M\rangle_s\right)^{\frac{p}{2}}\right] \le E\left[\sup_{t\le\tau}\left|\int_0^t H_s\, dM_s\right|^p\right] \le C_p E\left[\left(\int_0^\tau H_s^2\, d\langle M\rangle_s\right)^{\frac{p}{2}}\right].$$

b) Let $M \in \mathcal{M}^c_{0,\text{loc}}$ and let H be a predictable process such that for each $T \ge 0$,

$$E\left[\sqrt{\int_0^T H_s^2\, d\langle M\rangle_s}\right] < \infty.$$

Show that $H \in L^2_{\text{loc}}(M)$ and that the stochastic integral $\int H\, dM$ is a martingale.
Hint: First, show the general fact that if $N = (N_t)_{t\ge 0}$ is a local martingale and Y is an integrable random variable such that $|N_t| \le Y$ for all $t \ge 0$, then N is a uniformly integrable martingale.

c) Deduce from b) that if for each $T \ge 0$, $E[\sup_{t\le T} |M_t|] < \infty$ and the stopped process H^T is bounded, then $\int H\, dM$ is a martingale.
Hint: Use part a) first.

4. Matlab Exercise Let $(B_s)_{0\le s\le 1}$ be a standard Brownian motion on $[0,1]$. The aim of this exercise is to approximate[1] the process $(Y_t)_{0\le t\le 1} = (\int_0^t B_s dB_s)_{0\le t\le 1}$ with

$$\sum_{t_i \le t, t_i\in\Pi_n} B_{t_i}\,(B_{t_{i+1}\wedge t} - B_{t_i}),$$

[1]Recall from the lecture that for any continuous semimartingale X, any adapted RCLL process H and any sequence of partitions $(\Pi_n)_{n\in\mathbb{N}}$ of $[0,\infty)$ with $\lim\limits_{n\to\infty} |\Pi_n| = 0$, we have

$$\int_0^t H_{s-}\, dX_s = \lim_{n\to\infty} \sum_{t_i\le t, t_i\in\Pi_n} H_{t_i}\,(X_{t_{i+1}\wedge t} - X_{t_i}) \quad \text{in probability.}$$

Siehe nächstes Blatt!

where Π_n is a partition on $[0, 1]$. Here, we use an equidistant grid on $[0, 1]$, i.e., $0 = t_0 < t_1 < t_2 \ldots < t_n = 1$ with $n = 10^3$, $h = 10^{-3}$ and $t_i = i \cdot h$. Plot one sample path of the approximation and compare it with the exact solution Y. Also compute the L_2-norm of the error at terminal time, i.e., $\|Y_1^n - Y_1\|_{L^2}$, where Y^n denotes the approximated solution.

Hint: The exact solution can be computed by applying Itô's formula.