

Brownian Motion and Stochastic Calculus

Sketch of Solution Sheet 11

- 1. a)** We see that the SDE is of the form

$$dX_t = a(X_t) dt + b(X_t) dB_t, \quad X_0 = x \in \mathbb{R}.$$

where

$$a(x) = \sqrt{1+x^2} + \frac{x}{2} \quad \text{and} \quad b(x) = \sqrt{1+x^2}.$$

We observe that

$$\sup_{x \in \mathbb{R}} |b'(x)| = \sup_{x \in \mathbb{R}} \left| \frac{x}{\sqrt{1+x^2}} \right| \leq 1$$

as well as

$$\sup_{x \in \mathbb{R}} |a'(x)| = \sup_{x \in \mathbb{R}} \left| \frac{x}{\sqrt{1+x^2}} + \frac{1}{2} \right| \leq \frac{3}{2}.$$

Thus, from the differential mean value theorem, we obtain for $K := \frac{5}{2}$ that $a(\cdot)$ and $b(\cdot)$ satisfy the Lipschitz condition

$$|a(y) - a(z)| + |b(y) - b(z)| \leq K|y - z|, \quad y, z \in \mathbb{R}.$$

Moreover, we observe that for any $x \in \mathbb{R}$,

$$\begin{aligned} \left| \sqrt{1+x^2} + \frac{x}{2} \right| &\leq \left| 1 + |x| + \frac{x}{2} \right| \leq \frac{3}{2} (1 + |x|), \\ |\sqrt{1+x^2}| &\leq 1 + |x|. \end{aligned}$$

Thus we get for any $x \in \mathbb{R}$ the existence of a unique strong solution directly from Theorem 4.7.4 in the lecture notes.

- b)** We consider the function $f(x) := \operatorname{arsinh} x \in C^2$ (i.e. the inverse function of the hyperbolic sine). Thus, we obtain that

$$f'(x) = \frac{1}{\sqrt{1+x^2}} \quad \text{and} \quad f''(x) = -\frac{x}{(1+x^2)^{3/2}}.$$

Thus, applying Itô's formula to $Y_t := f(X_t)$, we obtain that

$$dY_t = df(X_t) = dt + dB_t, \quad Y_0 = \operatorname{arsinh} x.$$

Bitte wenden!

which implies that

$$X_t = \sinh Y_t = \sinh(\operatorname{arsinh} x + t + B_t), \quad t \geq 0.$$

- 2. a)** Consider the function $f(x, t) = xe^{\lambda t}$. Itô's formula applied to f yields

$$\begin{aligned} f(X_t, t) &= X_0 e^{\lambda 0} + \int_0^t X_s \lambda e^{\lambda s} ds + \int_0^t e^{\lambda s} dX_s \\ &= X_0 e^{\lambda 0} + \int_0^t X_s \lambda e^{\lambda s} ds + \int_0^t e^{\lambda s} \lambda (\nu - X_s) ds + \int_0^t e^{\lambda s} \sigma dW_s \\ &= X_0 e^{\lambda 0} + \int_0^t e^{\lambda s} \lambda \nu ds + \int_0^t e^{\lambda s} \sigma dW_s \\ &= X_0 e^{\lambda 0} + \nu(e^{\lambda t} - 1) + \int_0^t e^{\lambda s} \sigma dW_s. \end{aligned}$$

Now multiplying both sides by $e^{-\lambda t}$ and inserting the initial value $X_0 = x$ \mathbb{P} -a.s. gives

$$f(X_t, t)e^{-\lambda t} = X_t = xe^{-\lambda t} + \nu(1 - e^{-\lambda t}) + \int_0^t e^{\lambda(s-t)} \sigma dW_s, \quad t \geq 0.$$

- b)** To compute $\mathbb{E}[X_t]$ we first show that $(\int_0^t \sigma e^{\lambda(s-t)} dW_s)_{0 \leq t \leq T}$ is a $(\mathbb{P}, \mathcal{F})$ -martingale on $[0, T]$ with mean 0. We notice that the integrand is continuous and adapted (and hence predictable and locally bounded). Therefore, the stochastic integral is a local-martingale. Since

$$\mathbb{E} \left[\int_0^T \sigma^2 e^{2\lambda(s-T)} ds \right] = \frac{\sigma^2}{2\lambda} (1 - e^{-2\lambda T}) < \infty,$$

it is even a true martingale by EX 7-2. Thus,

$$\mathbb{E}[X_T] = xe^{-\lambda T} + \nu(1 - e^{-\lambda T}).$$

Moreover, using Itô's isometry we have

$$\begin{aligned} \text{Var}[X_T] &= \mathbb{E}[(X_T - \mathbb{E}[X_T])^2] \\ &= \mathbb{E} \left[\left(\sigma \int_0^T e^{\lambda(s-T)} dW_s \right)^2 \right] \\ &= \sigma^2 \mathbb{E} \left[\int_0^T e^{2\lambda(s-T)} ds \right] \\ &= \frac{\sigma^2}{2\lambda} (1 - e^{-2\lambda T}). \end{aligned}$$

Siehe nächstes Blatt!

c) Note that Itô's formula implies

$$\begin{aligned} dY_t &= 2X_t(-\lambda X_t dt + \sigma dW_t) + \sigma^2 dt \\ &= (-2\lambda X_t^2 + \sigma^2) dt + 2\sigma X_t dW_t \\ &= (-2\lambda Y_t + \sigma^2) dt + 2\sigma \sqrt{Y_t} \text{sign}(X_t) dW_t. \end{aligned}$$

Therefore, it suffices to note that $B_t := \int_0^t \text{sign}(X_s) dW_s$ defines a Brownian motion by Lévy's theorem, where $\text{sgn}(x)$ denotes the sign function, i.e., $\text{sgn}(x) = 1$ if $x > 0$ and $\text{sgn}(x) = -1$ if $x \leq 0$.

3. a) By contradiction, suppose that X has a strong solution. Since X is \mathbb{F} adapted we have $\mathbb{F}(X) \subseteq \mathbb{F} = \mathbb{F}(W)$ where $\mathbb{F}(X)$ denotes the filtration generated by X . Moreover, since $\text{sgn}(X_t)$ is adapted and left continuous, X is a continuous local (P, \mathbb{F}) -martingale null at 0 with

$$[X]_t = \int_0^t (\text{sgn}(X_s))^2 d[W]_t = t.$$

Therefore, by Lévy's theorem X is even a (P, \mathbb{F}) -Brownian motion. By definition, we have

$$W_t = \int_0^t (\text{sgn}(X_s))^2 dW_s = \int_0^t \text{sgn}(X_s) dX_s.$$

Using Tanaka's formula we see that W is adapted to $\mathbb{F}(|X|)$. Hence, we have $\mathbb{F}(X) \subseteq \mathbb{F} = \mathbb{F}(W) \subseteq \mathbb{F}(|X|)$ which is clearly a contradiction.

- b) To find a weak solution let Q be the Wiener measure on the path space $\Omega = C[0, \infty)$ and X be the coordinate process such that X is a Q -BM. Moreover, let \mathbb{F} be the (augmented) canonical filtration and define W as

$$W := \int \text{sgn}(X) dX.$$

As before, using Lévy's theorem W is a (Q, \mathbb{F}) -BM. Therefore,

$$\text{sgn}(X_t) dW_t = (\text{sgn}(X_t))^2 dX_t = dX_t.$$

4. Matlab Files

```
1 function bmsc114a
2 % In this exercise we simulated N paths of a OU process
3 % dX_t = \lambda (\nu - X_t) dt + \sigma dW_t, X_0 = x and
4 % simulate the
4 % expectation of X_I, X_I^2, X_I^+
```

Bitte wenden!

```

5 tic
6 %% parameter input
7 % horizon
8 T=1;
9 % sample size
10 Nsimu=10^5;
11 Nplot=Nsimu;
12 % grid points
13 M=10^3;
14 % volatility
15 sigma=0.3;
16 lambda=1;
17 nu=1.2;
18 x=1;
19 % time step
20 dt= T/M;
21
22 % theoretical value for the expectation , second moment
23 % and pos part
23 multilde = x*exp(-lambda*T)+ nu*(1-exp(-lambda*T));
24 sigmatilde= sqrt(sigma^2/(2*lambda)*(1-exp(-2*lambda*T)))
25 theoreticalvalueexp= multilde;
26 theoreticalvaluesec= sigmatilde^2+multilde^2;
27 theoreticalvaluepos= multilde*normcdf(multilde/sigmatilde)
28 % Simulation
29 % BM
30 BM = [ zeros(1,Nplot); sqrt(T/M)*cumsum(randn(M,Nplot)) ];
31 OU = [ x*ones(1,Nplot); zeros(M,Nplot) ];
32 % the process X
33 for i =1:M
34     OU(i+1,:)=OU(i,:)+lambda*(nu-OU(i,:))*dt+ sigma.* (BM
35     (i+1,:)-BM(i,:));
36 end
37 %plot the first 10 sample paths
38 timegrid= 0:dt:T;
39 plot(timegrid ,OU(:,1:10))
40
41 %compute simulated value

```

Siehe nächstes Blatt!

```

42 simulatedvalueexp= mean(OU(end,:));
43 simulatedvaluesec= mean(OU(end,:).^2);
44 simulatedvaluepos= mean(subplus(OU(end,:)));
45
46 disp('Exact values: Expectation/2.Moment/ pos. part')
47 disp([theoreticalvalueexp;theoreticalvaluesec;
        theoreticalvaluepos])
48 disp('Estimated value: Expectation/2.Moment/ pos. part')
49 disp([simulatedvalueexp;simulatedvaluesec;
        simulatedvaluepos])
50
51 %estimated variance
52 %estvarexp= var(OU(end,:));
53 %estvarsec= var(OU(end,:).^2);
54 %estvarpos= var(subplus(OU(end,:)));
55 % confidence interval using CLT
56 %cfplusexp=simulatedvalueexp+1.96*sqrt(estvarexp/Nsimu);
57 %cfminusexp=simulatedvalueexp-1.96*sqrt(estvarexp/Nsimu)
      ;
58 %cfplussec=simulatedvaluesec+1.96*sqrt(estvarsec/Nsimu);
59 %cfminussec=simulatedvaluesec-1.96*sqrt(estvarsec/Nsimu)
      ;
60 %cfpluspos=simulatedvaluepos+1.96*sqrt(estvarpos/Nsimu);
61 %cfminuspos=simulatedvaluepos-1.96*sqrt(estvarpos/Nsimu)
      ;
62
63 %disp('Confidence interval: ')
64 %disp([cfminusexp,cfplusexp; cfminussec,cfplussec;
        cfminuspos,cfpluspos])
65
66 toc

```

```

1 function bmsc114b
2 % In this exercise we simulated N paths of a CIR process
3 %  $dX_t = \lambda(\nu - X_t) dt + \sigma \sqrt{X_t} dW_t$ ,
   X_0 = x and simulate the
4 % expectation of  $X_1$ ,  $X_1^2$ ,  $X_1^+$ 
5 tic
6 %% parameter input
7 % horizon
8 T=1;
9 % sample size
10 Nsimu=10^5;

```

Bitte wenden!

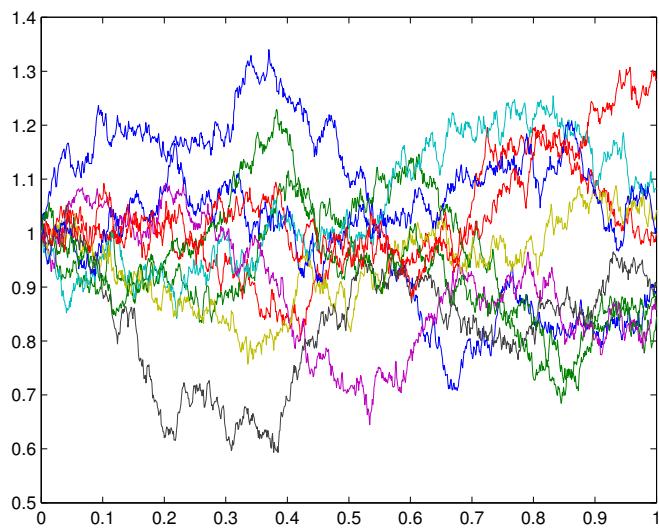


Abbildung 1: 10 sample paths of a OU process

```

11 Nplot=Nsimu;
12 % grid points
13 M=10^3;
14 % volatility
15 lambda=1;
16 nu=1.2;
17 sigma=0.3;
18 x=1;
19 % time step
20 dt= T/M;
21
22 % Check the Feller condition
23 check = 2*lambda*nu >= sigma^2;
24 if check == 1
25     disp('Feller condition is assumed')
26 elseif check==0
27     disp('Feller condition is not satisfied')
28 end
29
30 %% Some theoretical results about Y
31 % theoretical value for the expectation , second moment
32 constc = 4*lambda/(sigma^2*(1-exp(-lambda*T)));
33 constv = 4*lambda*nu/(sigma^2);

```

Siehe nächstes Blatt!

```

34 constlambda= constc*x*exp(-lambda*T);
35 % constc_T*X_T is non central chi square distributed (
% constv , constlambda)
36 % E(non-central chisquare distribution) = constv+
% constlambda;
37 %theoreticalvalueexp= (constv+constlambda)/constc;
38 theoreticalvalueexp= x*exp(-lambda*T)+nu*(1-exp(-lambda*
T));
39 % Var(non-central chisquare distribution) = 2(constv+2*
constlambda)
40 %theoreticalvaluesec= (2*(constv+2*constlambda)+(constv+
constlambda)^2)/(constc^2);
41 theoreticalvaluesec= x*sigma^2/lambda*(exp(-lambda*T)-
exp(-2*lambda*T))+nu*sigma^2/(2*lambda)*(1-exp(-lambda*
T))^2+theoreticalvalueexp^2;
42 % Use numerical integration to obtain a theoretical
value
43 % Since the Feller condition is assumed, E[X^+_1] must
be equal to E[X_1]
44 theoreticalvaluepos=integral(@(x) x.*ncx2pdf(x,constv,
constlambda),0,Inf)/constc;
45 %% Simulation
46 % BM
47 BM = [zeros(1,Nplot);sqrt(T/M)*cumsum(randn(M,Nplot))];
48
49 % Be careful, since the support of the normal
distribution is unbounded, we
50 % might get negative values for the CIR process, if the
parameters are
51 % chosen very "close to zero".
52 CIR = [x*ones(1,Nplot);zeros(M,Nplot)];
53 % the process Y
54 for i =1:M
55     CIR(i+1,:)=CIR(i,:)+lambda*(nu-CIR(i,:))*dt+ sigma.*%
sqrt(CIR(i,:)).*(BM(i+1,:)-BM(i,:));
56 end
57
58 %plot the first 10 sample paths
59 timegrid= 0:dt:T;
60 plot(timegrid,CIR(:,1:10))
61
62 %compute simulated value

```

Bitte wenden!

```

63 simulatedvalueexp= mean(CIR(end,:));
64 simulatedvaluesec= mean(CIR(end,:).^2);
65 simulatedvaluepos= mean(subplus(CIR(end,:)));
66
67 disp('Exact values: Expectation/2.Moment/pos. part')
68 disp([theoreticalvalueexp;theoreticalvaluesec;
    theoreticalvaluepos])
69 disp('Estimated value: Expectation/2.Moment/pos. part')
70 disp([simulatedvalueexp;simulatedvaluesec;
    simulatedvaluepos])
71
72 %estimated variance
73 %estvarexp= var(CIR(end,:));
74 %estvarsec= var(CIR(end,:).^2);
75 %estvarpos= var(subplus(CIR(end,:)));
76 % confidence interval using CLT
77 %cfplusexp=simulatedvalueexp+1.96*sqrt(estvarexp/Nsimu);
78 %cfminusexp=simulatedvalueexp-1.96*sqrt(estvarexp/Nsimu)
    ;
79 %cfplussec=simulatedvaluesec+1.96*sqrt(estvarsec/Nsimu);
80 %cfminussec=simulatedvaluesec-1.96*sqrt(estvarsec/Nsimu)
    ;
81 %cfpluspos=simulatedvaluepos+1.96*sqrt(estvarpos/Nsimu);
82 %cfminuspos=simulatedvaluepos-1.96*sqrt(estvarpos/Nsimu)
    ;
83
84 %disp('Confidence interval: ')
85 %disp([cfminusexp,cfplusexp; cfminussec,cfplussec;
    cfminuspos,cfpluspos])
86
87 toc

```

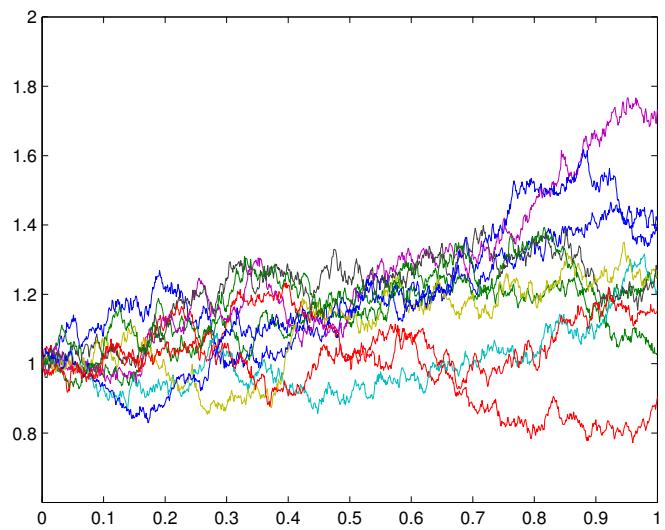


Abbildung 2: 10 sample paths of a CIR process